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A SYMPLECTIC DISCONTINUOUS GALERKIN FULL DISCRETIZATION FOR STOCHASTIC MAXWELL EQUATIONS*

CHUCHU CHEN†

Abstract. This paper proposes a fully discrete method called the symplectic discontinuous Galerkin (dG) full discretization for stochastic Maxwell equations driven by additive noises, based on a stochastic symplectic method in time and a dG method with the upwind fluxes in space. A priori H^k-regularity ($k \in \{1,2\}$) estimates for the solution of stochastic Maxwell equations are presented, which have not been reported before to the best of our knowledge. These H^k-regularities are vital to making the assumptions of the mean-square convergence analysis on the initial fields, the noise, and the medium coefficients, but not on the solution itself. The convergence order of the symplectic dG full discretization is shown to be $k/2$ in the temporal direction and $k-1/2$ in the spatial direction. Meanwhile we reveal the small noise asymptotic behaviors of the exact and numerical solutions via the large deviation principle, and show that the fully discrete method preserves the divergence relations in a weak sense.

Key words. stochastic Maxwell equations, symplectic dG full discretization, mean-square convergence

AMS subject classifications. 60H15, 35Q61

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1. Introduction.

Stochastic Maxwell equations are often used to better understand the role of thermodynamic fluctuations presented in the electromagnetic fields, and to get a deeper insight regarding the propagation of electromagnetic waves in complex media (see, e.g., [16]). A mathematically rigorous framework on the effects of randomness has been developed in [15]. The numerical treatment of the three dimensional stochastic Maxwell equations, even in the linear case, is a challenging task, due to the interaction of the large scale and the randomness of the problem. In this paper, we first discretize stochastic Maxwell equations in time via the midpoint scheme, which inherits the stochastic symplecticity of the original continuous problem, and subsequently in space based on a discontinuous Galerkin (dG) method combining its attractive features on the treatment of complex geometries and composite media.

For the time-dependent stochastic Maxwell equations, there exist some works on the construction of full discretizations, for example, multisymplectic numerical methods (cf. [7, 12]), energy-conserving methods (cf. [13]). On the rigorous error analysis of the numerical approximations, the existing works mainly focus on the temporal semidiscretizations (see [5, 6, 8]). It is shown in [5] that a semi-implicit Euler scheme converges with order 1/2 in the mean-square sense, and in [8] that the exponential integrators have mean-square convergence order 1/2, when applied to stochastic Maxwell equation with multiplicative Itô noise. Authors in [6] show that the stochastic symplectic Runge–Kutta semidiscretizations are mean-square convergent with

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†LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100149, China (chenchuchu@lsec.cc.ac.cn).

order 1 in the additive case. As far as we know, there are few works on the rigorous error analysis of the spatio-temporal full discretizations for the time-dependent stochastic Maxwell equations. The difficulty lies in the lack of regularity of the solution in H^k-norms or even in C^k-norms, which depends on the spatial domain, the medium coefficients, and the noise, etc. For example, on a cuboid, the solution of the time-harmonic deterministic Maxwell equations only has H^α-regularity for $\alpha < 3$ in general.

In this work, we consider the approximation of the stochastic electric and magnetic fileds $\mathbf{E}(t,x)$ and $\mathbf{H}(t,x)$ satisfying the following stochastic Maxwell equations on a cuboid $D = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+) \subset \mathbb{R}^3$,

$$\varepsilon \mathrm{d}\mathbf{E} - \nabla \times \mathbf{H}\mathrm{d}t = -\mathrm{d}W_e(t), \qquad (t,\mathbf{x}) \in (0,\ T] \times D, \tag{1.1a}$$

$$\mu \mathrm{d}\mathbf{H} + \nabla \times \mathbf{E}\mathrm{d}t = -\mathrm{d}W_m(t), \qquad (t,\mathbf{x}) \in (0,\ T] \times D, \tag{1.1b}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0,\ \nabla \cdot (\mu \mathbf{H}) = 0, \qquad (t,\mathbf{x}) \in (0,\ T] \times D, \tag{1.1c}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0},\ \mathbf{n} \cdot (\mu \mathbf{H}) = 0, \qquad (t,\mathbf{x}) \in (0,\ T] \times \partial D, \tag{1.1d}$$

$$\mathbf{E}(0,\mathbf{x}) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0(\mathbf{x}), \qquad \mathbf{x} \in D, \tag{1.1e}$$

where $T > 0$, and $\mathbf{n}(\mathbf{x})$ denotes the outer unit normal at $\mathbf{x} \in \partial D$. We suppose that the medium is isotropic, which implies that the permittivity ε and the permeability μ are real-valued scalar functions, i.e., $\varepsilon, \mu : D \to \mathbb{R}$. Throughout this paper, we assume the medium coefficients satisfy

$$\varepsilon,\ \mu \in L^\infty(D), \quad \varepsilon,\ \mu \geq \delta \text{ for a constant } \delta > 0. \tag{1.2}$$

Here $W_e(t)$ (resp., $W_m(t)$) is a Q_e-Wiener (resp., Q_m-Wiener) process with respect to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ with Q_e (resp., Q_m) being a symmetric, positive definite operator with finite trace on $U = L^2(D)^3$. Moreover, $W_e(t)$ and $W_m(t)$ are independent. The phase flow of (1.1) preserves the stochastic symplecticity (cf. [6]), i.e., if ε, μ are constants, for any $t \in [0,T]$, $\overline{\omega}(t) = \int_D d\mathbf{E}(t,\mathbf{x}) \wedge d\mathbf{H}(t,\mathbf{x})\mathrm{d}\mathbf{x} = \overline{\omega}(0)$, $\mathbb{P}$-a.s.

The solution theory of (1.1), which is crucial in the mean-square error analysis, is presented in section 2 with certain assumptions being made on the medium coefficients, the initial fields, and the noise. We restrict the Maxwell operator M on the closed subspace $\mathbb{V}_0$ of $\mathbb{V} := L^2(D)^3 \times L^2(D)^3$, in order to respect all boundary conditions and divergence properties. These conditions and properties are important to get the $L^p(\Omega; C([0,T]; H^1(D)^6))$-regularity ($H^1$-regularity in short) for the solution of (1.1), under the first order regularity and certain compatibility conditions of the initial data and the noise term; see Proposition 2.1. Furthermore, we can guarantee that the solution has H^2-regularity if more assumptions on the medium coefficients, the initial fields, and the noise are employed; see Proposition 2.2.

In order to inherit the stochastic symplectic structure, we apply the midpoint scheme (3.1) to discretize (1.1) in time in section 3. The error is measured in $L^2(\Omega; \mathbb{V})$, and gives a bound of order $k/2$ provided that the solutions of the continuous problem (1.1) and the temporal semidiscretization (3.1) belong to the domain $\mathcal{D}(M^k)$ of the kth power of the Maxwell operator M with $k \in \{1,2\}$ (see section 2 for notations). It is also shown that the divergence conservation laws (1.1c) are preserved numerically by the semidiscretization (3.1) in time.

We discretize the temporal semidiscretization (3.1) further in space using a dG method, and then it results in the fully discrete method (5.1), called the symplectic dG full discretization; see also section 4 for the treatment of the dG approximation of stochastic Maxwell equations. We refer interested readers to [17] for the application of the dG method to the time-harmonic stochastic Maxwell equations with colored noise, to [3] for its application to the stochastic Helmholtz-type equation, to [1] for its application to the stochastic Allen–Cahn equation, to [2] for its application to the semilinear stochastic wave equation, to [14] for its application to stochastic conservation laws, and to [4] for the application of a symplectic local dG method to the stochastic Schrödinger equation. Since the highest regularity of stochastic Maxwell equations that can be guaranteed is in H^2, the dG space is taken to be the set of piecewise linear functions. The upwind fluxes are utilized, due to the higher convergence order than the central fluxes; see [11] for the deterministic case. It is shown in Theorem 4.9 that the mean-square convergence order of the dG approximation (4.4) is of $k-1/2$ if the exact solution of (1.1) belongs to $L^p(\Omega; C([0,T]; H^k(D)^6))$ with $k \in \{1,2\}$. This convergence analysis is presented in a form applied also to the full discretization (5.1), which is stated in section 5. We also show that the divergence properties (1.1c) are preserved numerically in a weak sense by the spatial semidiscretization (4.4) and the full discretization (5.1) in Propositions 4.7 and 5.1, respectively. Moreover, the asymptotic behaviors of the exact and numerical solutions of stochastic Maxwell equations with small noise are investigated in section 2 and sections 3–5 (including the case for the temporal semidiscretization, the spatial semidiscretization, and the full discretization), respectively.

To conclude, the main contribution of this paper is to provide a rigorous error analysis of a full discretization for stochastic Maxwell equations. In particular, we prove that

(i) the exact solution and the numerical solution of the temporally semidiscrete method belong to $L^p(\Omega; C([0,T]; H^k(D)^6))$ with $k \in \{1,2\}$ depending only on the assumptions on the medium coefficients, the initial fields, and the noise, which have not been reported before to the best of our knowledge;

(ii) the mean-square error of the full discretization in $L^2(\Omega; \mathbb{V})$ is of order $k/2$ in time and of order $k-1/2$ in space ($k \in \{1,2\}$), which retains the convergence order of the upwind fluxes space discretization in the deterministic case.

2. Properties of stochastic Maxwell equations. This section presents the notations and basic results for stochastic Maxwell equations, including the stochastic symplectic structure, the regularity in $L^p(\Omega; C([0,T]; H^k(D)^6))$ with $k \in \{1,2\}$, and the small noise asymptotic behavior. Throughout this paper, we use C to denote a generic constant, independent of the step sizes τ and h, which may differ from line to line. Let $\Gamma_j^{\pm}$ be the open faces of D given by $x_j = a_j^{\pm}$, respectively, for $j = 1,2,3$.

2.1. Preliminaries. We first collect notations used throughout this paper. We use the standard Sobolev spaces $W^{k,p}(D) := W^{k,p}(D,\mathbb{R})$ for $k \in \mathbb{N}$, $p \in [1,\infty]$, where we denote $H^k(D) := W^{k,2}(D)$. For a real number $\gamma \in (0,1)$ and a normed real vector space V, denote by $C^\gamma([0,T]; V) := \{f : [0,T] \to V \text{ with } \|f\|_{C^\gamma([0,T];V)} < \infty\}$ the space of all γ-Hölder continuous functions from $[0,T]$ to V, where

$$\|f\|_{C^\gamma([0,T];V)} := \sup_{t\in[0,T]} \|f(t)\|_V + \sup_{t_1,t_2\in[0,T],t_1\neq t_2} \frac{\|f(t_2)-f(t_1)\|_V}{|t_2-t_1|^\gamma}.$$

Stochastic Maxwell equations (1.1) are studied in the real Hilbert space $\mathbb{V} = L^2(D)^3 \times L^2(D)^3$, endowed with the inner product

$$\left\langle \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{H}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix} \right\rangle_{\mathbb{V}} = \int_D (\varepsilon \mathbf{E}_1 \cdot \mathbf{E}_2 + \mu \mathbf{H}_1 \cdot \mathbf{H}_2) \mathrm{d}\mathbf{x}$$

for all $(\mathbf{E}_1^\top, \mathbf{H}_1^\top)^\top,\ (\mathbf{E}_2^\top, \mathbf{H}_2^\top)^\top \in \mathbb{V}$, and the norm

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\mathbb{V}} = \left[\int_D \left(\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 \right) \mathrm{d}\mathbf{x} \right]^{1/2} \quad \forall\ (\mathbf{E}^\top, \mathbf{H}^\top)^\top \in \mathbb{V}.$$

This space $\mathbb{V}$ is equivalent to the usual $L^2(D)^6$ space under the assumption (1.2) on coefficients ε and μ.

In addition we use the Hilbert spaces

$$H(\mathrm{curl}, D) := \{ v \in L^2(D)^3 :\ \nabla \times v \in L^2(D)^3 \},$$
$$H_0(\mathrm{curl}, D) := \{ v \in H(\mathrm{curl}, D) :\ \mathbf{n} \times v|_{\partial D} = \mathbf{0} \},$$

endowed with the norm

$$\|u\|_{\mathrm{curl}}^2 = \|u\|_{L^2(D)^3}^2 + \|\nabla \times u\|_{L^2(D)^3}^2,$$

and

$$H(\mathrm{div}, D) := \{ v \in L^2(D)^3 :\ \nabla \cdot v \in L^2(D) \},$$
$$H_0(\mathrm{div}, D) := \{ v \in H(\mathrm{div}, D) :\ \mathbf{n} \cdot v|_{\partial D} = 0 \},$$

endowed with the norm

$$\|u\|_{\mathrm{div}}^2 = \|u\|_{L^2(D)^3}^2 + \|\nabla \cdot u\|_{L^2(D)}^2.$$

After these preparations we introduce the Maxwell operator

$$M = \begin{pmatrix} 0 & \varepsilon^{-1} \nabla \times \\ -\mu^{-1} \nabla \times & 0 \end{pmatrix}, \quad \mathcal{D}(M) = H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D) \tag{2.1}$$

on $\mathbb{V}$. By defining $u(t) = (\mathbf{E}(t)^\top, \mathbf{H}(t)^\top)^\top$, the system (1.1) can be rewritten as a stochastic evolution equation

$$\begin{cases} \mathrm{d}u(t) = Mu(t)\mathrm{d}t - \mathrm{d}W(t), \\ u(0) = u_0, \end{cases} \tag{2.2}$$

where $W(t) = (\varepsilon^{-1} W_e(t)^\top,\ \mu^{-1} W_m(t)^\top)^\top$ is a Q-Wiener process on $\mathbb{V}$ with

$$Q = \begin{pmatrix} \varepsilon^{-1} Q_e & 0 \\ 0 & \mu^{-1} Q_m \end{pmatrix}.$$

In fact, for any $a = (a_1^\top, a_2^\top)^\top$, $b = (b_1^\top, b_2^\top)^\top \in \mathbb{V}$, we have

$$\begin{aligned} &\mathbb{E}\left[\langle W(t), a \rangle_{\mathbb{V}} \langle W(t), b \rangle_{\mathbb{V}} \right] \\ &= \mathbb{E}\left[\left(\langle W_e(t), a_1 \rangle_U + \langle W_m(t), a_2 \rangle_U \right) \left(\langle W_e(t), b_1 \rangle_U + \langle W_m(t), b_2 \rangle_U \right) \right] \\ &= t \langle Q_e a_1, b_1 \rangle_U + t \langle Q_m a_2, b_2 \rangle_U = t \langle Q a, b \rangle_{\mathbb{V}}. \end{aligned}$$

Note that $\mathbb{E}\|W(t)\|_{\mathbb{V}}^2 = t\big(\|\varepsilon^{-\frac{1}{2}} Q_e^{\frac{1}{2}}\|_{HS(U,U)}^2 + \|\mu^{-\frac{1}{2}} Q_m^{\frac{1}{2}}\|_{HS(U,U)}^2 \big)$, and Q still is a symmetric, positive definite operator on $\mathbb{V}$ with trace $\mathrm{Tr}(Q) = (\|\varepsilon^{-\frac{1}{2}} Q_e^{\frac{1}{2}}\|_{HS(U,U)}^2 +$

$\|\mu^{-\frac{1}{2}}Q_m^{\frac{1}{2}}\|^2_{HS(U,U)}$). Here $HS(U,H)$ denotes the space of all Hilbert–Schmidt operators from one separable Hilbert space U to another separable Hilbert space H, equipped with the inner product $\langle\Gamma_1,\Gamma_2\rangle_{HS(U,H)}=\sum_{j=1}^{\infty}\langle\Gamma_1\eta_j,\Gamma_2\eta_j\rangle_H$ and the norm $\|\Gamma\|_{HS(U,H)}=(\sum_{j=1}^{\infty}\|\Gamma\eta_j\|_H^2)^{1/2}$, where $\{\eta_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of U. It is not difficult to show that the energy of the system (1.1) evolves with a rate $\mathrm{Tr}(Q)$, i.e., $\mathbb{E}\|u(t)\|^2_{\mathbb{V}}=\mathbb{E}\|u_0\|^2_{\mathbb{V}}+\mathrm{Tr}(Q)t$.

Note that (2.2) is an infinite dimensional Hamiltonian system. If the coefficients ε,μ are constants, the canonical form of the infinite dimensional Hamiltonian system of (2.2) reads as

$$\mathrm{d}u(t)=\mathbb{J}^{-1}\frac{\delta\mathcal{H}}{\delta u}\mathrm{d}t+\mathbb{J}^{-1}\frac{\delta\mathcal{H}_1}{\delta u}\mathrm{d}\tilde{W}_e+\mathbb{J}^{-1}\frac{\delta\mathcal{H}_2}{\delta u}\mathrm{d}\tilde{W}_m, \tag{2.3}$$

where

$$\mathbb{J}=\begin{pmatrix}0 & I_3\\ -I_3 & 0\end{pmatrix}$$

with I_3 being the identity matrix on $\mathbb{R}^{3\times3}$, $\tilde{W}_e=(\mathbf{0}^\top,W_e^\top)^\top$, $\tilde{W}_m=(W_m^\top,\mathbf{0}^\top)^\top$, and $\mathcal{H}=-\frac{1}{2}\int_D\left(\mu^{-1}\mathbf{E}\cdot(\nabla\times\mathbf{E})+\varepsilon^{-1}\mathbf{H}\cdot(\nabla\times\mathbf{H})\right)\mathrm{d}\mathbf{x}$, $\mathcal{H}_1=\int_D\varepsilon^{-1}\mathbf{H}\mathrm{d}\mathbf{x}$, $\mathcal{H}_2=-\int_D\mu^{-1}\mathbf{E}\mathrm{d}\mathbf{x}$. The phase flow of (2.3) preserves the stochastic symplecticity, i.e., for any $t\in[0,T]$, $\overline{\omega}(t)=\int_D d\mathbf{E}(t,\mathbf{x})\wedge d\mathbf{H}(t,\mathbf{x})\mathrm{d}\mathbf{x}$, $\mathbb{P}$-a.s. We refer to [6] for the discussion on the symplecticity of stochastic Maxwell equations and the numerical preservation of the symplecticity by these semidiscrete methods in time.

The domain $\mathcal{D}(M)$ includes the electric boundary condition, but neither the magnetic boundary condition nor the divergence conditions. In order to regard all conditions, we define $\mathbb{V}_0:=\{(\mathbf{E}^\top,\mathbf{H}^\top)^\top\in\mathbb{V}:\ \nabla\cdot(\varepsilon\mathbf{E})=\nabla\cdot(\mu\mathbf{H})=0,\ \mathbf{n}\cdot(\mu\mathbf{H})=0\text{ on }\partial D\}$, which is a closed subspace of $\mathbb{V}$ with the inner product and norm being defined the same as in $\mathbb{V}$. We mainly work with the restriction M_0 of M on $\mathbb{V}_0$. It is known that under (1.2), $M_0:\mathcal{D}(M_0)=\mathcal{D}(M)\cap\mathbb{V}_0\to\mathbb{V}_0$ is skew adjoint, and thus generates a unitary C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ on $\mathbb{V}_0$. Denote by $\mathcal{D}(M^k):=\{u\in\mathcal{D}(M^{k-1}):M^{k-1}u\in\mathcal{D}(M)\}$ the domain of the kth power of M for $k\in\mathbb{N}_0:=\mathbb{N}\backslash\{0\}$ with norm $\|u\|_{\mathcal{D}(M^k)}:=\left(\|u\|_H^2+\|M^ku\|_H^2\right)^{1/2}$. Since M maps $\mathcal{D}(M)$ into $\mathbb{V}_0$, we have $\mathcal{D}(M_0^k)=\mathcal{D}(M^k)\cap\mathbb{V}_0$ (cf. [10]).

2.2. H^1-regularity. The H^1-regularity of the solution is deduced by utilizing the fact that $v\in H(\mathrm{curl},D)\cap H(\mathrm{div},D)$ belongs to $H^1(D)^3$ if $v\times\mathbf{n}=\mathbf{0}$ or $v\cdot\mathbf{n}=0$ holds on ∂D. Moreover, the H^1-norm of v is dominated by $\|v\|_{H^1(D)^3}\le C\left(\|v\|_{L^2(D)^3}+\|\nabla\times v\|_{L^2(D)^3}+\|\nabla\cdot v\|_{L^2(D)}\right)$, where the constant C depends on the space domain D. Since $\nabla\cdot(\varepsilon\mathbf{E})=0$, we get that $\nabla\cdot\mathbf{E}=\nabla\cdot\left(\varepsilon^{-1}\varepsilon\mathbf{E}\right)=\varepsilon^{-1}\nabla\cdot(\varepsilon\mathbf{E})+\nabla(\varepsilon^{-1})\cdot(\varepsilon\mathbf{E})=-\varepsilon^{-1}\nabla\varepsilon\cdot\mathbf{E}$ belongs to $L^2(D)^3$ if $\varepsilon\in W^{1,\infty}(D)$ with $\varepsilon\ge\delta$ for a constant $\delta>0$, and analogously for $\mathbf{H}$. That means that $\|\nabla\cdot\mathbf{E}\|_{L^2(D)}+\|\nabla\cdot\mathbf{H}\|_{L^2(D)}\le C\left(\delta,\|\varepsilon\|_{W^{1,\infty}(D)},\|\mu\|_{W^{1,\infty}(D)}\right)\|(\mathbf{E},\mathbf{H})\|_{L^2(D)^6}$. Hence, $\mathcal{D}(M_0)=\mathcal{D}(M)\cap\mathbb{V}_0\hookrightarrow H^1(D)^6$, if coefficients ε,μ satisfy the assumptions above. Moreover,

$$\|(\mathbf{E},\mathbf{H})\|_{H^1(D)^6}\le C\|(\mathbf{E},\mathbf{H})\|_{\mathcal{D}(M_0)} \tag{2.4}$$

with $C:=C\left(\delta,\|\varepsilon\|_{W^{1,\infty}(D)},\|\mu\|_{W^{1,\infty}(D)}\right)$.

Proposition 2.1. *Let the assumption* (1.2) *hold, and let $Q^{\frac{1}{2}}\in HS(\mathbb{V},\ \mathcal{D}(M_0))$ and $u_0\in L^p(\Omega;\mathcal{D}(M_0))$ for some $p\ge2$. Then* (2.2) *has a unique solution $u\in L^p(\Omega;\ C([0,T];\mathcal{D}(M_0)))$ given by*

$$u(t)=S(t)u_0-\int_0^t S(t-s)\mathrm{d}W(s), \tag{2.5}$$

where u also belongs to $C^{\frac{1}{2}}([0,T];\ L^p(\Omega;\ \mathbb{V}_0))$. Assume further that $\varepsilon,\ \mu\in W^{1,\infty}(D)$, then

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}\|u(t)\|^p_{H^1(D)^6}\bigg]\le C\mathbb{E}\bigg[\sup_{t\in[0,T]}\|u(t)\|^p_{\mathcal{D}(M_0)}\bigg]\le C(1+\mathbb{E}\|u_0\|^p_{\mathcal{D}(M_0)}), \tag{2.6}$$

where C depends on T, δ, $\|\varepsilon\|_{W^{1,\infty}(D)}$, $\|\mu\|_{W^{1,\infty}(D)}$, and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\ \mathcal{D}(M_0))}$.

Proof. Since M_0 generates a unitary C_0-group $\{S(t)\}_{t\in\mathbb{R}}$ on $\mathbb{V}_0$, the existence and uniqueness of the mild solution $u(t)$ of (2.5) on $\mathbb{V}_0$ follows. The estimate on stochastic convolution yields

$$\begin{aligned}&\Big[\mathbb{E}\big(\sup_{t\in[0,T]}\|u(t)\|^p_{\mathcal{D}(M_0)}\big)\Big]^{\frac{1}{p}}\le\Big[\mathbb{E}\big(\|u_0\|^p_{\mathcal{D}(M_0)}\big)\Big]^{\frac{1}{p}}\\ &\quad+\bigg[\mathbb{E}\bigg(\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)\mathrm{d}W(s)\Big\|^p_{\mathcal{D}(M_0)}\bigg)\bigg]^{\frac{1}{p}}\le C\Big(1+\Big[\mathbb{E}\big(\|u_0\|^p_{\mathcal{D}(M_0)}\big)\Big]^{\frac{1}{p}}\Big),\end{aligned} \tag{2.7}$$

where the constant C depends on T and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\ \mathcal{D}(M_0))}$.

Based on [5, Lemma 3.3] and (2.7), for any $0\le s\le t\le T$, we get

$$\begin{aligned}\|u(t)-u(s)\|_{L^p(\Omega;\mathbb{V}_0)}&\le\|\big(S(t-s)-I\big)u(s)\|_{L^p(\Omega;\mathbb{V}_0)}+\bigg\|\int_s^t S(t-r)\mathrm{d}W(r)\bigg\|_{L^p(\Omega;\mathbb{V}_0)}\\ &\le C(1+\|u_0\|_{L^p(\Omega;\mathcal{D}(M))})(t-s)+C(t-s)^{\frac{1}{2}},\end{aligned}$$

which leads to

$$\|u\|_{C^{\frac{1}{2}}([0,T];\ L^p(\Omega;\ \mathbb{V}_0))}=\sup_{t\in[0,T]}\|u(t)\|_{L^p(\Omega;\ \mathbb{V}_0)}+\sup_{t\neq s}\frac{\|u(t)-u(s)\|_{L^p(\Omega;\ \mathbb{V}_0)}}{|t-s|^{\frac{1}{2}}}\le C.$$

Utilizing the embedding (2.4), the assertion (2.6) follows from (2.7). ∎

2.3. H^2-regularity. In our error analysis we need the solution u of (2.2) taking values in $H^2(D)^6$, which relies on additional regularity properties of $\mathcal{D}(M_0^2)=\mathcal{D}(M^2)\cap\mathbb{V}_0$ and some smoothness of coefficients ε and μ. Assume that

$$\varepsilon,\mu\in W^{1,\infty}(D)\cap W^{2,3}(D)\quad\text{with } \varepsilon,\mu\ge\delta \text{ for a constant } \delta>0. \tag{2.8}$$

In fact, for any $w=(\mathbf{E}^\top,\mathbf{H}^\top)^\top\in\mathcal{D}(M_0^2)$, we already have $w\in H^1(D)^6$ from (2.4). Further,

$$M_0^2w=\begin{pmatrix}-\varepsilon^{-1}\nabla\times\big(\mu^{-1}\nabla\times\mathbf{E}\big)\\ -\mu^{-1}\nabla\times\big(\varepsilon^{-1}\nabla\times\mathbf{H}\big)\end{pmatrix}\in L^2(D)^6,$$

and the properties of the curl operator lead to

$$\begin{aligned}\Delta\mathbf{E}&=-\nabla\times(\nabla\times\mathbf{E})+\nabla(\nabla\cdot\mathbf{E})\\ &=-\mu\nabla\times(\mu^{-1}\nabla\times\mathbf{E})-\mu^{-1}\nabla\mu\times\big(\nabla\times\mathbf{E}\big)-\nabla(\varepsilon^{-1}\nabla\varepsilon\cdot\mathbf{E})\in L^2(D)^3,\end{aligned}$$

if coefficients $\varepsilon,\ \mu$ satisfy (2.8). Then the H^2-regularity of $\mathbf{E}$ follows from the equivalence of the H^2-norm and the graph norm of Laplacian Δ on D under certain mixed boundary conditions, i.e., if there is a unique function $v\in H^1_\Gamma(D)$ solving

$$\int_D v\phi\mathrm{d}\mathbf{x}+\int_D\nabla v\cdot\nabla\phi\mathrm{d}\mathbf{x}=\int_D f\phi\mathrm{d}\mathbf{x}$$

for $f \in L^2(D)$ and $\forall \phi \in H^1_\Gamma(D)$, then the solution $v \in H^2(D) \cap H^1_\Gamma(D)$ satisfies $v - \Delta v = f$ on D, $\partial_{\mathbf{n}} v = 0$ on $\partial D \backslash \Gamma$, and $\|v\|_{H^2(D)} \leq C\left(\|v\|_{L^2(D)} + \|\Delta v\|_{L^2(D)}\right)$ with the constant C depending on D. Here for a union $\Gamma \subseteq \partial D$ of some faces of D, $H^1_\Gamma(D) := \{v \in H^1(D) | \ \operatorname{tr}(v) = 0 \text{ on } \Gamma\}$. For each component E_j (resp., H_j) of $\mathbf{E}$ (resp., $\mathbf{H}$), the boundary Γ may be taken as $\Gamma^{\pm}_k \cup \Gamma^{\pm}_\ell$ (resp., $\Gamma^{\pm}_j$) with $j, k, \ell \in \{1, 2, 3\}$ and $k \neq \ell \neq j$. We refer to [10] for more details.

PROPOSITION 2.2. *Let $Q^{\frac{1}{2}} \in HS(\mathbb{V}, \mathcal{D}(M_0^2))$ and $u_0 \in L^p(\Omega; \mathcal{D}(M_0^2))$ for some $p \geq 2$. Under the assumption* (2.8)*, the solution* (2.5) *has the following property,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|^p_{H^2(D)^6}\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} \|u(t)\|^p_{\mathcal{D}(M_0^2)}\right] \leq C(1 + \mathbb{E}\|u_0\|^p_{\mathcal{D}(M_0^2)}), \tag{2.9}$$

where C depends on T, δ, $\|\varepsilon\|_{W^{1,\infty}(D)}$, $\|\varepsilon\|_{W^{2,3}(D)}$, $\|\mu\|_{W^{1,\infty}(D)}$, $\|\mu\|_{W^{2,3}(D)}$, and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\, \mathcal{D}(M_0^2))}$.

Proof. We first prove the $\mathcal{D}(M_0^2)$-regularity of the solution. From (2.5), we get

$$\begin{aligned}\left[\mathbb{E}\big(\sup_{t\in[0,T]} \|u(t)\|^p_{\mathcal{D}(M_0^2)}\big)\right]^{\frac{1}{p}} &\leq \left[\mathbb{E}\big(\|u_0\|^p_{\mathcal{D}(M_0^2)}\big)\right]^{\frac{1}{p}} \\ &+ \left[\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \int_0^t S(t-s)\mathrm{d}W(s)\right\|^p_{\mathcal{D}(M_0^2)}\right)\right]^{\frac{1}{p}} \leq C\left(1 + \left[\mathbb{E}\big(\|u_0\|^p_{\mathcal{D}(M_0^2)}\big)\right]^{\frac{1}{p}}\right),\end{aligned} \tag{2.10}$$

where the constant C depends on T and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\, \mathcal{D}(M_0^2))}$.

The first inequality in (2.9) comes from the embedding $\mathcal{D}(M_0^2) \hookrightarrow H^2(D)^6$. Thus the proof is finished by combining (2.10). ∎

2.4. Small noise asymptotic behavior. We scale the noise in the system (2.2) by a small parameter $\sqrt{\lambda}$, $\lambda \in \mathbb{R}^+$, i.e.,

$$\begin{cases} \mathrm{d}u(t) = Mu(t)\mathrm{d}t - \sqrt{\lambda}\mathrm{d}W(t), \\ u(0) = u_0, \end{cases} \tag{2.11}$$

whose mild solution is given by $u^{u_0,\lambda}(t) = S(t)u_0 - \sqrt{\lambda}\int_0^t S(t-r)\mathrm{d}W(r)$. Denote the stochastic convolution by $W_M(t) = \int_0^t S(t-r)\mathrm{d}W(r)$. Then for arbitrary $T > 0$, $W_M(T)$ is Gaussian on $\mathbb{V}$ with mean 0 and covariance operator $Q_T := \operatorname{Cov}\big(W_M(T)\big) = \int_0^T S(r)QS^*(r)\mathrm{d}r$.

LEMMA 2.3 (see [9, Proposition 12.10]). *Assume that X is a Gaussian random variable with distribution $\mu = \mathcal{N}(0, \widetilde{Q})$ on a Hilbert space H. Then the family of random variables $\{X_\lambda := \sqrt{\lambda}X\}_{\lambda>0}$ (or measures $\{\mu_\lambda = \mathcal{L}(X_\lambda)\}_{\lambda>0}$) satisfies the large deviation principle with the good rate function*

$$I(x) = \begin{cases} \frac{1}{2}\|\widetilde{Q}^{-\frac{1}{2}}x\|^2_H, & x \in \widetilde{Q}^{\frac{1}{2}}(H), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.12}$$

where $\widetilde{Q}^{-\frac{1}{2}}$ is the pseudoinverse of $\widetilde{Q}^{\frac{1}{2}}$.

Based on Lemma 2.3, we get the following asymptotic behavior of the solution for (2.11) with a small diffusion coefficient, which states that the laws of solutions satisfy the large deviation principle with the good rate function (2.13).

Proposition 2.4. *For arbitrary $T > 0$ and $u_0 \in \mathbb{V}$, the family of distributions $\{\mathcal{L}(u^{u_0,\lambda}(T))\}_{\lambda>0}$ satisfies the large deviation principle with the good rate function*

$$I_T^{u_0}(v) = \begin{cases} \frac{1}{2}\|Q_T^{-\frac{1}{2}}(v - S(T)u_0)\|_{\mathbb{V}}^2, & v - S(T)u_0 \in Q_T^{\frac{1}{2}}(\mathbb{V}), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.13}$$

where $Q_T^{-\frac{1}{2}}$ is the pseudoinverse of $Q_T^{\frac{1}{2}}$.

Proof. We define a process $Y^\lambda(t) = u^{u_0,\lambda}(t) - S(t)u_0$, which satisfies (2.11) with initial data $Y^\lambda(0) = 0$. This means that $Y^\lambda(t) = -\sqrt{\lambda}W_M(t)$. Then by the large deviation principle for Gaussian measures (Lemma 2.3), it follows that the good rate function of $\{Y^\lambda(T)\}_{\lambda>0}$ is given by

$$I_T^0(v) = \begin{cases} \frac{1}{2}\|Q_T^{-\frac{1}{2}}v\|_{\mathbb{V}}^2, & v \in Q_T^{\frac{1}{2}}(\mathbb{V}), \\ +\infty, & \text{otherwise}. \end{cases} \tag{2.14}$$

In order to give the rate function of $\{u^{u_0,\lambda}(T)\}_{\lambda>0}$ based on (2.14), we use the definition of the large deviation principle. Let $A \in \mathcal{B}(\mathbb{V})$ be closed. Then $A - \{S(T)u_0\}$ still is closed in $\mathcal{B}(\mathbb{V})$ and hence

$$\begin{aligned} &\limsup_{\lambda\to 0}\left[\lambda \ln \mathbb{P}\{u^{u_0,\lambda}(T) \in A\}\right] = \limsup_{\lambda\to 0}\left[\lambda \ln \mathbb{P}\{Y^\lambda(T) \in A - \{S(T)u_0\}\}\right] \\ &\le -\inf_{\overline{v}\in A-\{S(T)u_0\}} I_T^0(\overline{v}) = -\inf_{v\in A} I_T^0(v - S(T)u_0) =: -\inf_{v\in A} I_T^{u_0}(v). \end{aligned}$$

In a similar way we can check that for any open $B \in \mathcal{B}(\mathbb{V})$,

$$\liminf_{\lambda\to 0}\left[\lambda \ln \mathbb{P}\{u^{u_0,\lambda}(T) \in B\}\right] \ge -\inf_{v\in B} I_T^0(v - S(T)u_0) = -\inf_{v\in B} I_T^{u_0}(v).$$

Since $I_T^{u_0}$ fulfills the same properties as I_T^0, i.e., $I_T^{u_0}$ is a good rate function, the proof is thus completed. ∎

Remark 2.5. If Q commutes with M, then $Q_T^{\frac{1}{2}}(\mathbb{V}) = Q^{\frac{1}{2}}(\mathbb{V})$. In fact, $Q_T = \int_0^T S(r)QS^*(r)\mathrm{d}r = TQ$.

3. Temporal semidiscretization by stochastic symplectic method. In this section, we study the semidiscretization in time of (2.2) by a midpoint scheme, which preserves the stochastic symplectic structure. The temporal semidiscretizations by a class of stochastic symplectic Runge–Kutta methods have been studied in [6]. It is shown in there that the methods are convergent with order one in the mean-square sense, if the solution has regularity in $\mathcal{D}(M^2)$.

For the time interval $[0, T]$, we introduce the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$. Let $\tau = T/N$, and $\Delta W^{n+1} = W(t_{n+1}) - W(t_n)$, $n = 0, 1, \ldots, N-1$. Applying the midpoint scheme to (2.2) in the temporal direction yields

$$u^{n+1} = u^n + \frac{\tau}{2}(Mu^n + Mu^{n+1}) - \Delta W^{n+1}, \tag{3.1}$$

which can also be written as

$$\varepsilon\mathbf{E}^{n+1} = \varepsilon\mathbf{E}^n + \frac{\tau}{2}(\nabla\times\mathbf{H}^n + \nabla\times\mathbf{H}^{n+1}) - \Delta W_e^{n+1}, \tag{3.2a}$$

$$\mu \mathbf{H}^{n+1} = \mu \mathbf{H}^n - \frac{\tau}{2}(\nabla \times \mathbf{E}^n + \nabla \times \mathbf{E}^{n+1}) - \Delta W_m^{n+1}. \tag{3.2b}$$

This scheme preserves the stochastic symplectic structure numerically, which is stated as follows.

PROPOSITION 3.1 (see [6, Theorem 4.3]). *Let ε, μ be constants. Under a zero boundary condition, the temporal semidiscretization* (3.1) *preserves the discrete stochastic symplectic structure $\overline{\omega}^{n+1} = \int_D d\mathbf{E}^{n+1} \wedge d\mathbf{H}^{n+1} \mathrm{d}\mathbf{x} = \int_D d\mathbf{E}^n \wedge d\mathbf{H}^n \mathrm{d}\mathbf{x} = \overline{\omega}^n$, $\mathbb{P}$-a.s.*

The divergence conservation laws (1.1c) can be preserved numerically by the temporal semidiscretization (3.1).

PROPOSITION 3.2. *For the temporal semidiscretization* (3.1), *if $Q^{\frac{1}{2}} v \in \mathbb{V}_0$ for any $v \in \mathbb{V}$, then for any $n = 0, 1, \ldots, N-1$,*

$$\nabla \cdot (\varepsilon \mathbf{E}^{n+1}) = \nabla \cdot (\varepsilon \mathbf{E}^n), \qquad \nabla \cdot (\mu \mathbf{H}^{n+1}) = \nabla \cdot (\mu \mathbf{H}^n), \qquad \mathbb{P}\text{-a.s.}$$

Proof. The proof follows from the identity $\nabla \cdot (\nabla \times U) = 0$ for $U: \mathbb{R}^3 \to \mathbb{R}^3$. ◻

The solution of the temporal semidiscretization (3.1) also has the same regularity as the exact solution of (2.2) by using embeddings $\mathcal{D}(M_0) \hookrightarrow H^1(D)^6$ and $\mathcal{D}(M_0^2) \hookrightarrow H^2(D)^6$. They are stated below without proof.

PROPOSITION 3.3. *Under the conditions of Proposition* 2.1, *the solution of the temporal semidiscretization* (3.1) *has regularity in $H^1(D)^6$, and*

$$\max_{0 \le n \le N} \mathbb{E}\|u^n\|^p_{H^1(D)^6} \le C(1 + \mathbb{E}\|u_0\|^p_{\mathcal{D}(M_0)}), \tag{3.3}$$

where C depends on T, δ, $\|\varepsilon\|_{W^{1,\infty}(D)}$, $\|\mu\|_{W^{1,\infty}(D)}$, and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\, \mathcal{D}(M_0))}$.

PROPOSITION 3.4. *Under the conditions of Proposition* 2.2, *the solution of the temporal semidiscretization* (3.1) *has regularity in $H^2(D)^6$, and*

$$\max_{0 \le n \le N} \mathbb{E}\|u^n\|^p_{H^2(D)^6} \le C(1 + \mathbb{E}\|u_0\|^p_{\mathcal{D}(M_0^2)}), \tag{3.4}$$

where C depends on T, δ, $\|\varepsilon\|_{W^{1,\infty}(D)}$, $\|\varepsilon\|_{W^{2,3}(D)}$, $\|\mu\|_{W^{1,\infty}(D)}$, $\|\mu\|_{W^{2,3}(D)}$, and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\, \mathcal{D}(M_0^2))}$.

Let $S_\tau = \left(I - \frac{\tau}{2}M\right)^{-1}\left(I + \frac{\tau}{2}M\right)$ and $T_\tau = \left(I - \frac{\tau}{2}M\right)^{-1}$. The mild version of (3.1) reads as

$$u^{n+1} = S_\tau u^n - T_\tau \Delta W^{n+1} = S_\tau^{n+1} u_0 - \sum_{j=1}^{n+1} S_\tau^{n+1-j} T_\tau \Delta W^j. \tag{3.5}$$

LEMMA 3.5. *There exists a positive constant C independent of τ such that $\|I - T_\tau\|_{\mathcal{L}(\mathcal{D}(M),\mathbb{V})} \le C\tau$.*

Proof. We define $\widetilde{v} = T_\tau v$ for any $v \in \mathcal{D}(M)$, which means that $\widetilde{v} = v + \frac{\tau}{2} M\widetilde{v}$. Taking the inner product with $\widetilde{v}$ yields $\frac{1}{2}[\|\widetilde{v}\|^2_{\mathbb{V}} - \|v\|^2_{\mathbb{V}} + \|\widetilde{v} - v\|^2_{\mathbb{V}}] = \frac{\tau}{2}\langle M\widetilde{v}, \widetilde{v}\rangle_{\mathbb{V}} = 0$. Hence $\|\widetilde{v}\|_{\mathbb{V}} = \|T_\tau v\|_{\mathbb{V}} \le \|v\|_{\mathbb{V}}$ leads to $\|T_\tau\|_{\mathcal{L}(\mathbb{V},\mathbb{V})} \le 1$.

The conclusion of this lemma is equivalent to $\|\widetilde{v} - v\|_{\mathbb{V}} \le C\tau \|v\|_{\mathcal{D}(M)}$. In fact, $\|\widetilde{v} - v\|_{\mathbb{V}} = \frac{\tau}{2}\|M\widetilde{v}\|_{\mathbb{V}} = \frac{\tau}{2}\|T_\tau M v\|_{\mathbb{V}} \le \frac{\tau}{2}\|v\|_{\mathcal{D}(M)}$. ◻

For the semigroups $S(t_n)$ and S_τ^n, we have the following estimates.

LEMMA 3.6. *For any integer $n \in \{1,\dots,N\}$, there exists a positive constant C independent of τ such that $\|S(t_n) - S_\tau^n\|_{\mathcal{L}(\mathcal{D}(M^k),\mathbb{V})} \le C\tau^{k/2}$ with $k \in \{1,2\}$.*

Proof. In order to estimate the error of semigroups, we denote $v(t) = S(t)v_0$ and $v^k = S_\tau^k v_0$. Then $\{v(t)\}_{t\in[0,T]}$ is the exact solution of $\frac{\mathrm{d}}{\mathrm{d}t}v = Mv,\ v(0) = v_0$, while $\{v^k\}_{0\le k\le N}$ is the solution of $v^k = v^{k-1} + \frac{\tau}{2}\big(Mv^{k-1} + Mv^k\big),\ v^0 = v_0$. Note that $v(t_k) = v(t_{k-1}) + \int_{t_{k-1}}^{t_k} Mv(s)\mathrm{d}s$ leads to

$$e^k = e^{k-1} + \frac{\tau}{2}\big(Me^{k-1} + Me^k\big) + \int_{t_{k-1}}^{t_k}\Big[Mv(s) - \frac{1}{2}Mv(t_{k-1}) - \frac{1}{2}Mv(t_k)\Big]\mathrm{d}s,$$

where $e^k = v(t_k) - v^k$. Applying $\langle\cdot,\ e^k + e^{k-1}\rangle_{\mathbb{V}}$ to both sides of the above equation, and using the skew-adjoint property of the operator M, we get

$$\begin{aligned}
\|e^k\|_{\mathbb{V}}^2 &= \|e^{k-1}\|_{\mathbb{V}}^2 + \int_{t_{k-1}}^{t_k}\Big\langle Mv(s) - \frac{1}{2}Mv(t_{k-1}) - \frac{1}{2}Mv(t_k),\ e^k + e^{k-1}\Big\rangle_{\mathbb{V}}\mathrm{d}s\\
&= \|e^{k-1}\|_{\mathbb{V}}^2 - \frac{1}{2}\int_{t_{k-1}}^{t_k}\Big\langle \int_{t_{k-1}}^{s} Mv(r)\mathrm{d}r - \int_{s}^{t_k} Mv(r)\mathrm{d}r,\ Me^k + Me^{k-1}\Big\rangle_{\mathbb{V}}\mathrm{d}s\\
&\le \|e^{k-1}\|_{\mathbb{V}}^2 + C\tau^2\Big(\sup_{t\in[0,T]}\|v(t)\|_{\mathcal{D}(M)}^2 + \max_{0\le k\le N}\|v^k\|_{\mathcal{D}(M)}^2\Big)\\
&\le \|e^{k-1}\|_{\mathbb{V}}^2 + C\tau^2\|v_0\|_{\mathcal{D}(M)}^2,
\end{aligned} \tag{3.6}$$

which yields $\max_{1\le k\le N}\|e^k\|_{\mathbb{V}} = \max_{1\le k\le N}\|\big(S(t_k) - S_\tau^k\big)v_0\|_{\mathbb{V}} \le C\tau^{\frac{1}{2}}\|v_0\|_{\mathcal{D}(M)}$.

On the other hand, based on (3.6),

$$\begin{aligned}
\|e^k\|_{\mathbb{V}}^2 &= \|e^{k-1}\|_{\mathbb{V}}^2 - \frac{1}{2}\int_{t_{k-1}}^{t_k}\Big\langle \int_{t_{k-1}}^{s} Mv(r)\mathrm{d}r - \int_{s}^{t_k} Mv(r)\mathrm{d}r,\ Me^k + Me^{k-1}\Big\rangle_{\mathbb{V}}\mathrm{d}s\\
&= \|e^{k-1}\|_{\mathbb{V}}^2 + \frac{1}{2}\int_{t_{k-1}}^{t_k}\Big\langle\Big(\int_{t_{k-1}}^{s}\int_{t_{k-1}}^{r} - \int_{s}^{t_k}\int_{t_{k-1}}^{r}\Big)Mv(\xi)\mathrm{d}\xi\mathrm{d}r,\ M^2(e^k + e^{k-1})\Big\rangle_{\mathbb{V}}\mathrm{d}s\\
&\le \|e^{k-1}\|_{\mathbb{V}}^2 + C\tau^3\|v_0\|_{\mathcal{D}(M^2)}^2,
\end{aligned}$$

which yields $\max_{1\le k\le N}\|e^k\|_{\mathbb{V}} = \max_{1\le k\le N}\|\big(S(t_k) - S_\tau^k\big)v_0\|_{\mathbb{V}} \le C\tau\|v_0\|_{\mathcal{D}(M^2)}$. ∎

THEOREM 3.7. *Let $Q^{\frac{1}{2}} \in HS(\mathbb{V},\mathcal{D}(M^k))$ and $u_0 \in L^2(\Omega;\mathcal{D}(M^k))$ with $k \in \{1,2\}$. For the temporal semidiscretization* (3.1), *we have*

$$\max_{1\le n\le N}\big(\mathbb{E}\|u(t_n) - u^n\|_{\mathbb{V}}^2\big)^{1/2} \le C\tau^{k/2} \quad \text{for } k \in \{1,2\}, \tag{3.7}$$

where C depends on T, $\|u_0\|_{L^2(\Omega;\mathcal{D}(M^k))}$, and $\|Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\mathcal{D}(M^k))}$, but is independent of τ and n.

Proof. From the mild solutions (2.5) and (3.5), we use the Itô isometry to get

$$\begin{aligned}
\mathbb{E}\|u(t_n) - u^n\|_{\mathbb{V}}^2 &\le 2\mathbb{E}\|\big(S(t_n) - S_\tau^n\big)u_0\|_{\mathbb{V}}^2 + 2\mathbb{E}\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\big(S(t_n - r) - S_\tau^{n-j}T_\tau\big)\mathrm{d}W\Big\|_{\mathbb{V}}^2\\
&= 2\mathbb{E}\|\big(S(t_n) - S_\tau^n\big)u_0\|_{\mathbb{V}}^2 + 2\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\Big\|\big(S(t_n - r) - S_\tau^{n-j}T_\tau\big)Q^{\frac{1}{2}}\Big\|_{HS(\mathbb{V},\mathbb{V})}^2\mathrm{d}r.
\end{aligned}$$

The first term on the right-hand side is estimated by Lemma 3.6, and the second term on the right-hand side can be estimated by, for $r \in [t_{j-1}, t_j]$,

$$\begin{aligned}\left\|\big(S(t_n - r) - S_\tau^{n-j} T_\tau\big) Q^{\frac12}\right\|_{HS(\mathbb{V},\mathbb{V})} &\le \|S(t_n - t_j)\big(S(t_j - r) - I\big) Q^{\frac12}\|_{HS(\mathbb{V},\mathbb{V})} \\ &\quad + \|\big(S(t_n - t_j) - S_\tau^{n-j}\big) Q^{\frac12}\|_{HS(\mathbb{V},\mathbb{V})} + \|S_\tau^{n-j}\big(I - T_\tau\big) Q^{\frac12}\|_{HS(\mathbb{V},\mathbb{V})} \\ &\le C\tau \|Q^{\frac12}\|_{HS(\mathbb{V},\mathcal{D}(M))} + C\tau^{k/2} \|Q^{\frac12}\|_{HS(\mathbb{V},\mathcal{D}(M^k))} \qquad \text{for } k \in \{1, 2\},\end{aligned}$$

where in the last step, we use Lemmas 3.5–3.6 and [5, Lemma 3.3]. Combining them, we finish the proof. ∎

Applying the midpoint scheme to discretize the system (2.11) with small noise, we get that $u^N = S_\tau^N u_0 - \sqrt{\lambda} \sum_{j=1}^N S_\tau^{N-j} T_\tau \Delta W^j$. Let $W_{M;N} := \sum_{j=1}^N S_\tau^{N-j} T_\tau \Delta W^j$. Then it is Gaussian on $\mathbb{V}$ with mean 0 and covariance operator $Q_{T;N} := \mathrm{Cov}(W_{M;N}) = \tau \sum_{j=1}^N \big(S_\tau^{N-j} T_\tau\big) Q \big(S_\tau^{N-j} T_\tau\big)^*$. Analogously, as in Proposition 2.4, we get the following result.

PROPOSITION 3.8. *For integer $N > 0$ and $u_0 \in \mathbb{V}$, the family of distributions $\{\mathcal{L}\big(u^{N;\, u_0, \lambda}\big)\}_{\lambda > 0}$ satisfies the large deviation principle with the good rate function*

$$I_{T,N}^{u_0}(v) = \begin{cases} \frac12 \|\big(Q_{T;N}\big)^{-\frac12}\big(v - S_\tau^N u_0\big)\|_{\mathbb{V}}^2, & v - S_\tau^N u_0 \in \big(Q_{T;N}\big)^{\frac12}(\mathbb{V}), \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.8}$$

Remark 3.9. If Q commutes with M, then $\big(Q_{T;N}\big)^{\frac12}(\mathbb{V}) = \big(T_\tau Q^{\frac12}\big)(\mathbb{V}) \subset Q^{\frac12}(\mathbb{V})$. In fact, $Q_{T;N} = \tau \sum_{j=1}^N \big(S_\tau^{N-j} T_\tau\big) Q \big(S_\tau^{N-j} T_\tau\big)^* = \tau N T_\tau Q T_\tau^* = T T_\tau Q T_\tau^*$ yields the assertion.

PROPOSITION 3.10. *Assume that Q commutes with M, and $v,\ u_0 \in \big(T_\tau Q^{\frac12}\big)(\mathbb{V})$. Then there is a constant C depending on T, $\|Q^{-\frac12} v\|_{\mathcal{D}(M)}$, and $\|Q^{-\frac12} u_0\|_{\mathcal{D}(M)}$ such that $|I_T^{u_0}(v) - I_{T,N}^{u_0}(v)| \le C\tau^{\frac12}$.*

In addition, if $Q^{-\frac12} v,\ Q^{-\frac12} u_0 \in \mathcal{D}(M^2)$, then there is a constant C depending on T, $\|Q^{-\frac12} v\|_{\mathcal{D}(M^2)}$, and $\|Q^{-\frac12} u_0\|_{\mathcal{D}(M^2)}$ such that $|I_T^{u_0}(v) - I_{T,N}^{u_0}(v)| \le C\tau$.

Proof. Note that under the conditions of this proposition,

$$I_T^{u_0}(v) = \frac{1}{2T} \left\|Q^{-\frac12}\left(v - S(T)u_0\right)\right\|_{\mathbb{V}}^2, \quad I_{T,N}^{u_0}(v) = \frac{1}{2T} \left\|Q^{-\frac12} T_\tau^{-1}\left(v - S_\tau^N u_0\right)\right\|_{\mathbb{V}}^2.$$

Thus,

$$\begin{aligned} &\left|I_T^{u_0}(v) - I_{T,N}^{u_0}(v)\right| \\ &= \frac{1}{2T}\Big|\Big\langle Q^{-\frac12}\left(v - S(T)u_0\right) + Q^{-\frac12} T_\tau^{-1}\left(v - S_\tau^N u_0\right), \\ &\qquad\qquad Q^{-\frac12}\left(v - S(T)u_0\right) - Q^{-\frac12} T_\tau^{-1}\left(v - S_\tau^N u_0\right)\Big\rangle_{\mathbb{V}}\Big| \\ &\le C\left\|Q^{-\frac12}\left(v - S(T)u_0\right) - Q^{-\frac12} T_\tau^{-1}\left(v - S_\tau^N u_0\right)\right\|_{\mathbb{V}} \\ &\le C\left[\left\|Q^{-\frac12}(I - T_\tau^{-1})\left(v - S(T)u_0\right)\right\|_{\mathbb{V}} + \left\|Q^{-\frac12} T_\tau^{-1}\left(S(T) - S_\tau^N\right) u_0\right\|_{\mathbb{V}}\right], \end{aligned} \tag{3.9}$$

where the constant C depends on T, $\|Q^{-\frac12} T_\tau^{-1} v\|_{\mathbb{V}}$, $\|Q^{-\frac12} T_\tau^{-1} u_0\|_{\mathbb{V}}$. Since $I - T_\tau^{-1} = \frac{\tau}{2} M$,

$$\left\|Q^{-\frac12}(I - T_\tau^{-1})\left(v - S(T)u_0\right)\right\|_{\mathbb{V}} \le C(\|Q^{-\frac12} M v\|_{\mathbb{V}},\ \|Q^{-\frac12} M u_0\|_{\mathbb{V}})\tau.$$

And for the second term on the right-hand side of (3.9),

$$\begin{aligned}
&\left\| Q^{-\frac{1}{2}} T_\tau^{-1} \left(S(T) - S_\tau^N \right) u_0 \right\|_{\mathbb{V}} \\
&\le \left\| Q^{-\frac{1}{2}} \left(S(T) - S_\tau^N \right) u_0 \right\|_{\mathbb{V}} + \frac{\tau}{2} \left\| Q^{-\frac{1}{2}} M \left(S(T) - S_\tau^N \right) u_0 \right\|_{\mathbb{V}} \\
&\le \left\| Q^{-\frac{1}{2}} \left(S(T) - S_\tau^N \right) u_0 \right\|_{\mathbb{V}} + C(\|Q^{-\frac{1}{2}} M u_0\|_{\mathbb{V}})\tau.
\end{aligned}$$

Lemma 3.6 yields the conclusion. ∎

4. Spatial semidiscretization by the dG method.

In this section, we investigate the semidiscretization of the stochastic Maxwell equations (2.2) in space by the dG method with the upwind fluxes, including the properties of the discrete Maxwell operator, the well-posedness of the spatial semidiscretization, the preservation of the divergence properties in a weak sense, and the mean-square error estimate of the semidiscrete method in space.

4.1. Discrete Maxwell operator.

The notations and properties of the discrete Maxwell operator are based on [11]. Let $\mathcal{T}_h = \{K\}$ be a simplicial, shape- and contact-regular mesh of the domain D consisting of elements K, i.e., $D = \bigcup K$. The index h refers to the maximum diameter of all elements of $\mathcal{T}_h$. The dG space with respect to the mesh $\mathcal{T}_h$ is taken to be the set of piecewise linear functions, i.e., $\mathbb{V}_h := \mathbb{P}_1(\mathcal{T}_h)^6 := \{v_h \in L^2(D) : \ v_h|_K \in \mathbb{P}_1(K)\}^6$, where $\mathbb{P}_1(K)$ denotes the set of continuous piecewise polynomials of degree ≤ 1. In general, $\mathbb{V}_h \not\subset \mathcal{D}(M_0)$. The set of faces is denoted by $\mathcal{G}_h = \mathcal{G}_h^{\text{int}} \cup \mathcal{G}_h^{\text{ext}}$, where $\mathcal{G}_h^{\text{int}}$ and $\mathcal{G}_h^{\text{ext}}$ consist of all interior and all exterior faces, respectively. By $\mathbf{n}_F$ we denote the unit normal of a face $F \in \mathcal{G}_h^{\text{int}}$, where the orientation of $\mathbf{n}_F$ is fixed once and forever for each inner face. And for a boundary face $F \in \mathcal{G}_h^{\text{ext}}$, $\mathbf{n}_F$ is an outward normal vector. The broken Sobolev spaces are defined by $H^k(\mathcal{T}_h) := \{v \in L^2(D) : \ v|_K \in H^k(K) \text{ for all } K \in \mathcal{T}_h\}$, $k \in \mathbb{N}$, with seminorm and norm being $|v|^2_{H^k(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)}$ and $\|v\|^2_{H^k(\mathcal{T}_h)} := \sum_{j=0}^k |v|^2_{H^j(\mathcal{T}_h)}$, respectively. Note that $H^k(D) \subset H^k(\mathcal{T}_h)$.

Assumption 4.1. Assume that $\pi_h : \ \mathbb{V} \to \mathbb{V}_h$ is the orthogonal projection, defined by, for every $v \in \mathbb{V}$,

$$\langle v - \pi_h v, \ u_h \rangle_{\mathbb{V}} = 0 \qquad \text{for all} \ \ u_h \in \mathbb{V}_h. \tag{4.1}$$

Moreover, for all $v \in H^s(\mathcal{T}_h)^6$ with integer $s \le 2$, it holds that

$$\|v - \pi_h v\|_{\mathbb{V}} \le C h^s |v|_{H^s(\mathcal{T}_h)^6} \tag{4.2}$$

and

$$\sum_{F\in\mathcal{G}_h} \|v - \pi_h v\|^2_{L^2(F)^6} \le C h^{2s-1} |v|^2_{H^s(\mathcal{T}_h)^6}, \tag{4.3}$$

where the constant C is independent of h.

Remark 4.2.

(i) For the projection operator π_h in Assumption 4.1, it is not difficult to get that $\|\pi_h v\|_{\mathbb{V}} \le \|v\|_{\mathbb{V}}$.

(ii) Suppose that $\mu_K := \mu|_K$ and $\varepsilon_K := \varepsilon|_K$ are constants for each $K \in \mathcal{T}_h$, then the usual L^2-orthogonal projection π_h on $\mathbb{P}_1(\mathcal{T}_h)$ satisfies Assumption 4.1, where the projection acts componentwise for vector fields.

Define by $[[v]]_F := (v_{K_F})|_F - (v_K)|_F$ the jump of v on an interior face F with normal vector $\mathbf{n}_F$ pointing from K to K_F. The Maxwell operator discretized by a dG method with the upwind fluxes is defined as follows.

Definition 4.3. *Given* $u_h = (\mathbf{E}_h^\top, \mathbf{H}_h^\top)^\top$, $v_h = (\psi_h^\top, \phi_h^\top)^\top \in \mathbb{V}_h$, *the discrete Maxwell operator* $M_h : \mathbb{V}_h \to \mathbb{V}_h$ *is given as*

$$
\begin{aligned}
&\langle M_h u_h, v_h\rangle_{\mathbb{V}} := \sum_K \Big(\langle \nabla\times \mathbf{H}_h, \psi_h\rangle_{L^2(K)^3} - \langle \nabla\times \mathbf{E}_h, \phi_h\rangle_{L^2(K)^3}\Big)\\
&+ \sum_{F\in\mathcal{G}_h^{\mathrm{int}}} \Big(\langle \mathbf{n}_F\times[[\mathbf{H}_h]]_F, \beta_K\psi_K + \beta_{K_F}\psi_{K_F}\rangle_{L^2(F)^3}\\
&- \langle \mathbf{n}_F\times[[\mathbf{E}_h]]_F, \alpha_K\phi_K + \alpha_{K_F}\phi_{K_F}\rangle_{L^2(F)^3}\\
&- \gamma_F\langle \mathbf{n}_F\times[[\mathbf{E}_h]]_F, \mathbf{n}_F\times[[\psi_h]]_F\rangle_{L^2(F)^3} - \delta_F\langle \mathbf{n}_F\times[[\mathbf{H}_h]]_F, \mathbf{n}_F\times[[\phi_h]]_F\rangle_{L^2(F)^3}\Big)\\
&+ \sum_{F\in\mathcal{G}_h^{\mathrm{ext}}} \Big(\langle \mathbf{n}_F\times\mathbf{E}_h, \phi_h\rangle_{L^2(F)^3} - 2\gamma_F\langle \mathbf{n}_F\times\mathbf{E}_h, \mathbf{n}_F\times\psi_h\rangle_{L^2(F)^3}\Big),
\end{aligned}
$$

where

$$
\alpha_K = \frac{C_{K_F}\varepsilon_{K_F}}{C_{K_F}\varepsilon_{K_F} + C_K\varepsilon_K}, \qquad \beta_K = \frac{C_{K_F}\mu_{K_F}}{C_{K_F}\mu_{K_F} + C_K\mu_K},
$$
$$
\gamma_F = \frac{1}{C_{K_F}\mu_{K_F} + C_K\mu_K}, \qquad \delta_F = \frac{1}{C_{K_F}\varepsilon_{K_F} + C_K\varepsilon_K}
$$

with $C_K = (\varepsilon_K\mu_K)^{-1/2}$.

The discrete Maxwell operator M_h is also well-defined as an operator from $\mathbb{V}_h + \big(\mathcal{D}(M)\cap H^1(\mathcal{T}_h)^6\big)$ to $\mathbb{V}_h$, and has the following properties. Here $\mathbb{V}_h + \big(\mathcal{D}(M)\cap H^1(\mathcal{T}_h)^6\big) := \{v_h + u :\ v_h\in\mathbb{V}_h,\ u\in\mathcal{D}(M)\cap H^1(\mathcal{T}_h)^6\}$. We refer to [11, Lemmas 4.3–4.5] for proofs.

Proposition 4.4.

(i) *For* $u\in\mathcal{D}(M)\cap H^1(\mathcal{T}_h)^6$, *we have* $M_h u = \pi_h M u$.

(ii) *For all* $u_h = (\mathbf{E}_h^\top, \mathbf{H}_h^\top)^\top \in \mathbb{V}_h$, *we have*

$$
\begin{aligned}
\langle M_h u_h, u_h\rangle_{\mathbb{V}} = &- \sum_{F\in\mathcal{G}_h^{\mathrm{int}}} \Big(\gamma_F\|\mathbf{n}_F\times[[\mathbf{E}_h]]_F\|^2_{L^2(F)^3} + \delta_F\|\mathbf{n}_F\times[[\mathbf{H}_h]]_F\|^2_{L^2(F)^3}\Big)\\
&- 2\sum_{F\in\mathcal{G}_h^{\mathrm{ext}}} \gamma_F\|\mathbf{n}_F\times\mathbf{E}_h\|^2_{L^2(F)^3} \le 0.
\end{aligned}
$$

In particular, M_h *is dissipative on* $\mathbb{V}_h$.

(iii) *For* $u = (\mathbf{E}^\top, \mathbf{H}^\top)^\top \in \mathbb{V}_h + \big(\mathcal{D}(M)\cap H^1(\mathcal{T}_h)^6\big)$ *and* $v_h = (\psi_h^\top, \phi_h^\top)^\top\in\mathbb{V}_h$, *we have*

$$
\begin{aligned}
&\langle M_h u, v_h\rangle_{\mathbb{V}} = \sum_K \Big(\langle \mathbf{H}, \nabla\times\psi_h\rangle_{L^2(K)^3} - \langle \mathbf{E}, \nabla\times\phi_h\rangle_{L^2(K)^3}\Big)\\
&+ \sum_{F\in\mathcal{G}_h^{\mathrm{int}}} \Big(\langle \beta_K\mathbf{H}_{K_F} + \beta_{K_F}\mathbf{H}_K - \gamma_F\mathbf{n}_F\times[[\mathbf{E}]]_F, \mathbf{n}_F\times[[\psi_h]]_F\rangle_{L^2(F)^3}\\
&\qquad - \langle \alpha_K\mathbf{E}_{K_F} + \alpha_{K_F}\mathbf{E}_K + \delta_F\mathbf{n}_F\times[[\mathbf{H}]]_F, \mathbf{n}_F\times[[\phi_h]]_F\rangle_{L^2(F)^3}\Big)\\
&- \sum_{F\in\mathcal{G}_h^{\mathrm{ext}}} \Big(\langle \mathbf{H}, \mathbf{n}_F\times\psi_h\rangle_{L^2(F)^3} + 2\gamma_F\langle \mathbf{n}_F\times\mathbf{E}, \mathbf{n}_F\times\psi_h\rangle_{L^2(F)^3}\Big).
\end{aligned}
$$

4.2. Semidiscrete method in space. After discretizing (2.2) by a dG method with the upwind fluxes, we end up with the spatial semidiscretization

$$\begin{cases} \mathrm{d}u_h(t) = M_h u_h(t)\mathrm{d}t - \pi_h \mathrm{d}W(t), \\ u_h(0) = \pi_h u_0, \end{cases} \tag{4.4}$$

where M_h is the discrete Maxwell operator in Definition 4.3, and $u_h(t) \in \mathbb{V}_h$ is an approximation of the exact solution $u(t) \in \mathbb{V}$.

Notice that (4.4) actually is a finite dimensional stochastic differential equation. In fact, let $\{\phi_1, \ldots, \phi_{N_h}\}$ be a basis for $\mathbb{V}_h$. Utilizing this basis, the semidiscrete problem (4.4) in space can be rewritten as, for $j = 1, \ldots, N_h$,

$$\begin{cases} \mathrm{d}\langle u_h(t), \phi_j\rangle_{\mathbb{V}} = \langle M_h u_h(t), \phi_j\rangle_{\mathbb{V}}\mathrm{d}t - \langle \phi_j, \mathrm{d}W(t)\rangle_{\mathbb{V}}, \\ \langle u_h(0), \phi_j\rangle_{\mathbb{V}} = \langle u_0, \phi_j\rangle_{\mathbb{V}}. \end{cases} \tag{4.5}$$

Since $u_h(t) \in L^2(\Omega; \mathbb{V}_h)$, we get $u_h(t) = \sum_{\ell=1}^{N_h} u_{[\ell]}(t)\phi_\ell$. Denoting $A = (\langle \phi_\ell, \phi_j\rangle_{\mathbb{V}})_{j,\ell} \in \mathbb{R}^{N_h\times N_h}$, $B = (\langle M_h\phi_\ell, \phi_j\rangle_{\mathbb{V}})_{j,\ell} \in \mathbb{R}^{N_h\times N_h}$, $\mathbf{u}(t) = (u_{[1]}(t), \ldots, u_{[N_h]}(t))^\top \in \mathbb{R}^{N_h}$, $\mathbf{u}_0 = (\langle u_0, \phi_1\rangle_{\mathbb{V}}, \ldots, \langle u_0, \phi_{N_h}\rangle_{\mathbb{V}})^\top \in \mathbb{R}^{N_h}$, and $\mathbf{W}(t) = (W_{[1]}(t), \ldots, W_{[N_h]}(t))^\top \in \mathbb{R}^{N_h}$ with $W_{[j]}(t) = \langle \phi_j, W(t)\rangle_{\mathbb{V}}$, we obtain the system of stochastic ordinary differential equations on $\mathbb{R}^{N_h}$ for (4.4):

$$\begin{cases} A\mathrm{d}\mathbf{u}(t) = B\mathbf{u}(t)\mathrm{d}t - \mathrm{d}\mathbf{W}(t), \\ A\mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{4.6}$$

Notice that the components of $\mathbf{W}(t)$ are correlated with

$$\mathbb{E}\big(W_{[j]}(t)W_{[\ell]}(t)\big) = \mathbb{E}\left(\langle \phi_j, W(t)\rangle_{\mathbb{V}}\langle \phi_\ell, W(t)\rangle_{\mathbb{V}}\right) = t\langle Q\phi_j, \phi_\ell\rangle_{\mathbb{V}} \quad \forall\, j, \ell = 1, \ldots, N_h.$$

PROPOSITION 4.5. *The spatially semidiscrete problem* (4.4) *is well-posed, i.e., there is a unique solution* $u_h \in L^2(\Omega; C([0,T]; \mathbb{V}_h))$ *given by*

$$u_h(t) = e^{tM_h}u_h(0) - \int_0^t e^{(t-s)M_h}\pi_h \mathrm{d}W(s). \tag{4.7}$$

Moreover, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u_h(t)\|_{\mathbb{V}}^2\Big] \le C(1 + \mathbb{E}\|u_0\|_{\mathbb{V}}^2), \tag{4.8}$$

where the constant C *depends on* T *and* $\mathrm{Tr}(Q)$.

Proof. Note that $I - M_h : \mathbb{V}_h \to \mathbb{V}_h$ is injective and surjective, and thus $\mathrm{Ran}(I - M_h) = \mathbb{V}_h$. Since the discrete operator M_h is dissipative on $\mathbb{V}_h$, it generates a contraction semigroup. Therefore, the unique solution of (4.4) is given by (4.7).

The estimate in (4.8) is obtained by the triangle inequality and the estimate on stochastic convolution

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]} \|u_h(t)\|_{\mathbb{V}}^2\Big] \\ &\le 2\mathbb{E}\Big[\sup_{t\in[0,T]} \|e^{tM_h}u_h(0)\|_{\mathbb{V}}^2\Big] + 2\mathbb{E}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t e^{(t-s)M_h}\pi_h \mathrm{d}W(s)\Big\|_{\mathbb{V}}^2\Big] \\ &\le 2\mathbb{E}\|u_h(0)\|_{\mathbb{V}}^2 + 2T\mathbb{E}\|\pi_h Q^{\frac{1}{2}}\|_{HS(\mathbb{V},\mathbb{V})}^2 \le C(1 + \mathbb{E}\|u_0\|_{\mathbb{V}}^2), \end{aligned}$$

where in the last step we use property $\|\pi_h u\|_{\mathbb{V}} \le \|u\|_{\mathbb{V}}$ of the projection operator. ∎

It is not difficult to observe that $W_{M;h}(t) = \int_0^t e^{(t-s)M_h}\pi_h \mathrm{d}W(s)$ is Gaussian on $\mathbb{V}_h$ with mean 0 and covariance operator

$$Q_{T,h} := \mathrm{Cov}\big(W_{M;h}(T)\big) = \int_0^T \big(e^{rM_h}\pi_h\big)Q\big(e^{rM_h}\pi_h\big)^*\mathrm{d}r.$$

Applying the dG method to discretize the spatial direction of the small noise system (2.11), we denote by $\{\mathcal{L}\big(u_h^{u_0,\lambda}(T)\big)\}_{\lambda>0}$ the laws of the semidiscrete solutions. The asymptotic behavior of $\{\mathcal{L}\big(u_h^{u_0,\lambda}(T)\big)\}_{\lambda>0}$ is similar to that of $\{\mathcal{L}\big(u^{u_0,\lambda}(T)\big)\}_{\lambda>0}$ in Proposition 2.4, which is stated below.

PROPOSITION 4.6. *For arbitrary $T>0$ and $u_0 \in \mathbb{V}$, the family of distributions $\{\mathcal{L}(u_h^{u_0,\lambda}(T))\}_{\lambda>0}$ satisfies the large deviation principle with the good rate function*

$$I_{T,h}^{u_0}(v) = \begin{cases} \frac{1}{2}\|Q_{T,h}^{-\frac{1}{2}}\big(v - e^{TM_h}\pi_h u_0\big)\|_{\mathbb{V}}^2, & v - e^{TM_h}\pi_h u_0 \in Q_{T,h}^{\frac{1}{2}}(\mathbb{V}_h), \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.9}$$

where $Q_{T,h}^{-\frac{1}{2}}$ is the pseudoinverse of $Q_{T,h}^{\frac{1}{2}}$.

4.3. Discrete divergence conservation property. If $u_0 \in \mathbb{V}_0$ and $Q^{\frac{1}{2}} \in HS(\mathbb{V},\mathbb{V}_0)$, the exact solution $u(t)$ of the stochastic Maxwell equations (2.2) possesses the divergence relations (1.1c): $\nabla\cdot(\varepsilon\mathbf{E}(t)) = 0$ and $\nabla\cdot(\varepsilon\mathbf{H}(t)) = 0$. However, for the spatial semidiscretization (4.4), we prove that the divergence relations are preserved numerically in the following discrete weak sense.

Define the test space $X_h \subset H_0^1(D)$ as $X_h := \{v \in C^0(\bar{D}) : v_h|_K \in \mathbb{P}_2(K),\ K \in \mathcal{T}_h\} \cap H_0^1(D)$. By $\langle\cdot,\cdot\rangle_{-1}$ we denote the duality product between $H^{-1}(D)$ and $H_0^1(D)$ in which $\langle \nabla\cdot E, \psi\rangle_{-1} = -\langle E, \nabla\psi\rangle_{L^2(D)^3}\ \forall\ E \in L^2(D)^3,\ \psi \in H_0^1(D)$.

PROPOSITION 4.7. *Let $u_0 \in \mathbb{V}_0$ and $Q^{\frac{1}{2}} \in HS(\mathbb{V},\mathbb{V}_0)$. The solution $(\mathbf{E}_h(t),\mathbf{H}_h(t))$ of the spatially semidiscrete problem (4.4) satisfies $\forall\ t\in[0,T]$, and $\forall\ \phi \in X_h$,*

$$\langle \nabla\cdot(\varepsilon\mathbf{E}_h(t)),\phi\rangle_{-1} = \langle \nabla\cdot(\mu\mathbf{H}_h(t)),\phi\rangle_{-1} = 0, \qquad \mathbb{P}\text{-}a.s.$$

Proof. For $\psi,\phi \in X_h$, using the definition of the duality product $\langle\cdot,\cdot\rangle_{-1}$, we get

$$\begin{aligned}\left\langle \begin{pmatrix}\nabla\cdot(\varepsilon\mathbf{E}_h(t))\\ \nabla\cdot(\mu\mathbf{H}_h(t))\end{pmatrix}, \begin{pmatrix}\psi\\ \phi\end{pmatrix}\right\rangle_{-1} &= \langle\nabla\cdot(\varepsilon\mathbf{E}_h(t)),\psi\rangle_{-1} + \langle\nabla\cdot(\varepsilon\mathbf{H}_h(t)),\phi\rangle_{-1}\\ &= -\langle\varepsilon\mathbf{E}_h(t),\nabla\psi\rangle_{L^2(D)^3} - \langle\varepsilon\mathbf{H}_h(t),\nabla\phi\rangle_{L^2(D)^3}\\ &= -\left\langle \begin{pmatrix}\mathbf{E}_h(t)\\ \mathbf{H}_h(t)\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix}\right\rangle_{\mathbb{V}}.\end{aligned}$$

Using (4.4) we obtain

$$\begin{aligned}\left\langle \begin{pmatrix}\mathbf{E}_h(t)\\ \mathbf{H}_h(t)\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix}\right\rangle_{\mathbb{V}} =& \left\langle \begin{pmatrix}\mathbf{E}_h(0)\\ \mathbf{H}_h(0)\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix}\right\rangle_{\mathbb{V}} + \int_0^t \left\langle M_h\begin{pmatrix}\mathbf{E}_h(s)\\ \mathbf{H}_h(s)\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix}\right\rangle_{\mathbb{V}}\mathrm{d}s\\ &- \left\langle \pi_h\begin{pmatrix}\varepsilon^{-1}W_e(t)\\ \mu^{-1}W_m(t)\end{pmatrix}, \begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix}\right\rangle_{\mathbb{V}}.\end{aligned}$$

For the first and third terms on the right-hand side, we utilize the property (4.1) of projection and the fact that

$$\begin{pmatrix}\nabla\psi\\ \nabla\phi\end{pmatrix} \in \mathbb{V}_h$$

to get

$$\left\langle \begin{pmatrix} \mathbf{E}_h(0) \\ \mathbf{H}_h(0) \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \left\langle \pi_h \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \left\langle \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = 0$$

and

$$\begin{aligned} \left\langle \pi_h \begin{pmatrix} \varepsilon^{-1} W_e(t) \\ \mu^{-1} W_m(t) \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} &= \left\langle \begin{pmatrix} \varepsilon^{-1} W_e(t) \\ \mu^{-1} W_m(t) \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} \\ &= \langle W_e(t), \nabla\psi \rangle_{L^2(D)^3} + \langle W_m(t), \nabla\phi \rangle_{L^2(D)^3} \\ &= -\langle \nabla \cdot W_e(t), \psi \rangle_{-1} - \langle \nabla \cdot W_m(t), \phi \rangle_{-1} = 0. \end{aligned}$$

Using Proposition 4.4(iii), the second term on the right-hand side equals zero, since for any function $\varphi \in X_h$, we have $\nabla \times \nabla\varphi = \mathbf{0}$, $\mathbf{n}_F \times [[\nabla\varphi]]_F = \mathbf{0}$ for $F \in \mathcal{G}_h^{\text{int}}$ and $\mathbf{n} \times \nabla\varphi = \mathbf{0}$ on ∂D. Therefore, the conclusion of this proposition comes from taking $\phi = 0$ or $\psi = 0$, respectively. ∎

Remark 4.8. The projection of the exact solution of (2.2) has the same property, $\forall\ t \in [0, T]$ and $\forall\ \phi \in X_h$,

$$\langle \nabla \cdot \pi_h(\varepsilon\mathbf{E}(t)), \phi \rangle_{-1} = \langle \nabla \cdot \pi_h(\mu\mathbf{H}(t)), \phi \rangle_{-1} = 0, \qquad \mathbb{P}\text{-a.s.}$$

In fact, since $\nabla\phi \in \mathbb{P}_1(\mathcal{T}_h)^3$, we have $\langle \nabla \cdot \pi_h(\varepsilon\mathbf{E}(t)), \phi \rangle_{-1} = \langle \pi_h(\varepsilon\mathbf{E}(t)), \nabla\phi \rangle_{L^2(D)^3} = \langle \varepsilon\mathbf{E}(t), \nabla\phi \rangle_{L^2(D)^3} = \langle \nabla \cdot (\varepsilon\mathbf{E}(t)), \phi \rangle_{-1} = 0$.

4.4. Error estimate of spatial semidiscretization. To investigate the error of the spatial semidiscretization (4.4), we apply the projection π_h to the continuous problem (2.2) and use Proposition 4.4(i) to get

$$\mathrm{d}\pi_h u(t) = M_h u(t)\mathrm{d}t - \pi_h \mathrm{d}W(t), \qquad \pi_h u(0) = \pi_h u_0. \tag{4.10}$$

We define the error $e(t) = u_h(t) - u(t) = \big(u_h(t) - \pi_h u(t)\big) - \big(u(t) - \pi_h u(t)\big) =: e_h(t) - e_\pi(t)$.

The mean-square error estimate of the spatial semidiscretization (4.4) is given in the following theorem.

THEOREM 4.9. *Let $u \in C([0, T]; L^2(\Omega; H^k(D)^6))$ with $k \in \{1, 2\}$ be the solution of (2.2) and let $u_h \in C([0, T]; L^2(\Omega; \mathbb{V}_h))$ be the solution of (4.4). Then there is a constant C independent of h such that $\sup_{t\in[0,T]}(\mathbb{E}\|u_h(t)-u(t)\|_{\mathbb{V}}^2)^{\frac{1}{2}} \le Ch^{k-\frac{1}{2}}$ for $k \in \{1, 2\}$.*

Proof. For the part $e_\pi(t)$, by using (4.2), we have

$$\mathbb{E}\|e_\pi(t)\|_{\mathbb{V}}^2 = \mathbb{E}\|u(t) - \pi_h u(t)\|_{\mathbb{V}}^2 \le Ch^{2k}\mathbb{E}|u(t)|_{H^k(D)^6}^2. \tag{4.11}$$

For the part $e_h(t)$, we subtract (4.10) from (4.4) to get $\mathrm{d}e_h(t) = M_h e_h(t)\mathrm{d}t - M_h e_\pi(t)\mathrm{d}t$, $e_h(0) = 0$. Then we obtain, for any $t \in [0, T]$,

$$\frac{1}{2}\|e_h(t)\|_{\mathbb{V}}^2 - \int_0^t \langle M_h e_h(s), e_h(s) \rangle_{\mathbb{V}} \mathrm{d}s = -\int_0^t \langle M_h e_\pi(s), e_h(s) \rangle_{\mathbb{V}} \mathrm{d}s.$$

For the term on the right-hand side, noticing $e_h(s) \in \mathbb{V}_h$, $e_\pi(s) \in \mathbb{V}_h + (\mathcal{D}(M) \cap H^1(D)^6)$, we use Proposition 4.4(iii) to obtain

$$\langle M_h e_\pi, e_h \rangle_{\mathbb{V}} = \sum_K \left(\langle e_\pi^{\mathbf{H}}, \nabla \times e_h^{\mathbf{E}} \rangle_{L^2(K)^3} - \langle e_\pi^{\mathbf{E}}, \nabla \times e_h^{\mathbf{H}} \rangle_{L^2(K)^3} \right)$$

$$+\sum_{F\in\mathcal{G}_h^{\mathrm{int}}}\Big(\langle\beta_K e^{\mathbf{H}}_{\pi,K_F}+\beta_{K_F}e^{\mathbf{H}}_{\pi,K}-\gamma_F\mathbf{n}_F\times[[e^{\mathbf{E}}_\pi]]_F,\mathbf{n}_F\times[[e^{\mathbf{E}}_h]]_F\rangle_{L^2(F)^3}$$
$$-\langle\alpha_K e^{\mathbf{E}}_{\pi,K_F}+\alpha_{K_F}e^{\mathbf{E}}_{\pi,K}+\delta_F\mathbf{n}_F\times[[e^{\mathbf{H}}_\pi]]_F,\mathbf{n}_F\times[[e^{\mathbf{H}}_h]]_F\rangle_{L^2(F)^3}\Big)$$
$$-\sum_{F\in\mathcal{G}_h^{\mathrm{ext}}}\Big(\langle e^{\mathbf{H}}_\pi,\mathbf{n}_F\times e^{\mathbf{E}}_h\rangle_{L^2(F)^3}+2\gamma_F\langle\mathbf{n}_F\times e^{\mathbf{E}}_\pi,\mathbf{n}_F\times e^{\mathbf{E}}_h\rangle_{L^2(F)^3}\Big),$$

where $e_\pi=\big((e^{\mathbf{E}}_\pi)^\top,\ (e^{\mathbf{H}}_\pi)^\top\big)^\top$ and $e_h=\big((e^{\mathbf{E}}_h)^\top,\ (e^{\mathbf{H}}_h)^\top\big)^\top$. The property of the projection π_h leads to $\langle e^{\mathbf{H}}_\pi,\nabla\times e^{\mathbf{E}}_h\rangle_{L^2(K)^3}=\langle e^{\mathbf{E}}_\pi,\nabla\times e^{\mathbf{H}}_h\rangle_{L^2(K)^3}=0$. Then using the Cauchy–Schwarz and Young's inequalities, we have

$$\begin{aligned}
\big|\langle M_h e_\pi(s),e_h(s)\rangle_{\mathbb{V}}\big| &\le \sum_{F\in\mathcal{G}_h^{ext}}\gamma_F\|\mathbf{n}_F\times e^{\mathbf{E}}_h\|^2_{L^2(F)^3}\\
&+\sum_{F\in\mathcal{G}_h^{int}}\Big(\frac{\gamma_F}{2}\|\mathbf{n}_F\times[[e^{\mathbf{E}}_h]]_F\|^2_{L^2(F)^3}+\frac{\delta_F}{2}\|\mathbf{n}_F\times[[e^{\mathbf{H}}_h]]_F\|^2_{L^2(F)^3}\Big)\\
&+\sum_{F\in\mathcal{G}_h^{int}}\Big(\frac{1}{2\gamma_F}\|\beta_K e^{\mathbf{H}}_{\pi,K_F}+\beta_{K_F}e^{\mathbf{H}}_{\pi,K}-\gamma_F\mathbf{n}_F\times[[e^{\mathbf{E}}_\pi]]_F\|^2_{L^2(F)^3}\\
&\qquad+\frac{1}{2\delta_F}\|\alpha_K e^{\mathbf{E}}_{\pi,K_F}+\alpha_{K_F}e^{\mathbf{E}}_{\pi,K}+\delta_F\mathbf{n}_F\times[[e^{\mathbf{H}}_\pi]]_F\|^2_{L^2(F)^3}\Big)\\
&+\sum_{F\in\mathcal{G}_h^{ext}}\Big(\frac{1}{2\gamma_F}\|e^{\mathbf{H}}_\pi\|^2_{L^2(F)^3}+2\gamma_F\|\mathbf{n}_F\times e^{\mathbf{E}}_\pi\|^2_{L^2(F)^3}\Big)\\
&\le -\frac12\langle M_h e_h(s),e_h(s)\rangle_{\mathbb{V}}+Ch^{2k-1}|u(s)|^2_{H^k(D)^6},
\end{aligned}\tag{4.12}$$

where in the last step, we use the equality in (ii) of Proposition 4.4 and the inequality (4.3). Hence, we have

$$\frac12\|e_h(t)\|^2_{\mathbb{V}}-\frac12\int_0^t\langle M_h e_h(s),e_h(s)\rangle_{\mathbb{V}}\mathrm{d}s\le Ch^{2k-1}\int_0^t|u(s)|^2_{H^k(D)^6}\mathrm{d}s.$$

Proposition 4.4(ii) yields that the second term on the left-hand side is nonnegative. Then taking the expectation and using Lemmas 2.1 and 2.2, we get $\sup_{t\in[0,T]}\mathbb{E}\|e_h(t)\|^2_{\mathbb{V}}\le Ch^{2k-1}\int_0^T\mathbb{E}|u(s)|^2_{H^k(D)^6}\mathrm{d}s$, which combined with (4.11) completes the proof. ∎

5. Full discretization of stochastic Maxwell equations.

In this section, we consider the full discretization of stochastic Maxwell equations (2.2) by applying the midpoint scheme in time and the dG method with the upwind fluxes in space:

$$u_h^{n+1}=u_h^n+\frac{\tau}{2}\left(M_h u_h^n+M_h u_h^{n+1}\right)-\pi_h\Delta W^{n+1}\tag{5.1}$$

with $u_h^0=\pi_h u_0$. Utilizing the basis of $\mathbb{V}_h$ in section 4, the fully discrete method (5.1) can be rewritten as the midpoint scheme for (4.6),

$$A\mathbf{u}^{n+1}=A\mathbf{u}^n+\frac{\tau}{2}\big(B\mathbf{u}^n+B\mathbf{u}^{n+1}\big)-\Delta\mathbf{W}^{n+1}.$$

Following the proof of Proposition 4.7, the divergence conservation property (1.1c) is preserved numerically by the solution of (5.1) in a weak sense.

PROPOSITION 5.1. *Let $u_0 \in \mathbb{V}_0$ and $Q^{\frac{1}{2}} \in HS(\mathbb{V}, \mathbb{V}_0)$. The solution $\{u_h^n\}_{0\le n\le N}$ of the fully discrete method* (5.1) *satisfies $\forall\ n \in \{0, 1, \ldots, N\}$ and $\forall\ \phi \in X_h$,*

$$\langle \nabla \cdot (\varepsilon \mathbf{E}_h^n), \phi\rangle_{-1} = \langle \nabla \cdot (\mu \mathbf{H}_h^n), \phi\rangle_{-1} = 0, \qquad \mathbb{P}\text{-}a.s.$$

Proof. For $\psi, \phi \in X_h$, using the definition of the inner product $\langle \cdot, \cdot\rangle_{-1}$, we get

$$\begin{aligned}\left\langle \begin{pmatrix} \nabla \cdot (\varepsilon \mathbf{E}_h^{n+1}) \\ \nabla \cdot (\mu \mathbf{H}_h^{n+1}) \end{pmatrix}, \begin{pmatrix} \psi \\ \phi \end{pmatrix} \right\rangle_{-1} &= \langle \nabla \cdot (\varepsilon \mathbf{E}_h^{n+1}), \psi\rangle_{-1} + \langle \nabla \cdot (\varepsilon \mathbf{H}_h^{n+1}), \phi\rangle_{-1} \\ &= -\left\langle \begin{pmatrix} \mathbf{E}_h^{n+1} \\ \mathbf{H}_h^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}}.\end{aligned}$$

Using (5.1) we obtain

$$\begin{aligned}\left\langle \begin{pmatrix} \mathbf{E}_h^{n+1} \\ \mathbf{H}_h^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} =& \left\langle \begin{pmatrix} \mathbf{E}_h^{n} \\ \mathbf{H}_h^{n} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} + \frac{\tau}{2} \left\langle M_h \begin{pmatrix} \mathbf{E}_h^{n} + \mathbf{E}_h^{n+1} \\ \mathbf{H}_h^{n} + \mathbf{H}_h^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} \\ &- \left\langle \pi_h \begin{pmatrix} \varepsilon^{-1}\Delta W_e^{n+1} \\ \mu^{-1}\Delta W_m^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}}.\end{aligned}$$

Using Proposition 4.4(iii), the second term on the right-hand side equals zero, since for any function $\varphi \in X_h$, we have $\nabla \times \nabla\varphi = \mathbf{0}$, $\mathbf{n}_F \times [[\nabla\varphi]]_F = \mathbf{0}$ for $F \in \mathcal{G}_h^{\text{int}}$, and $\mathbf{n} \times \nabla\varphi = \mathbf{0}$ on ∂D. For the third term on the right-hand side, the property of the projection (4.1), and the fact that

$$\begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \in \mathbb{V}_h$$

yield

$$\begin{aligned}\left\langle \pi_h \begin{pmatrix} \varepsilon^{-1}\Delta W_e^{n+1} \\ \mu^{-1}\Delta W_m^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} &= \left\langle \begin{pmatrix} \varepsilon^{-1}\Delta W_e^{n+1} \\ \mu^{-1}\Delta W_m^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} \\ &= -\langle \nabla \cdot (\Delta W_e^{n+1}), \psi\rangle_{-1} - \langle \nabla \cdot (\Delta W_m^{n+1}), \phi\rangle_{-1} = 0.\end{aligned}$$

Thus,

$$\left\langle \begin{pmatrix} \mathbf{E}_h^{n+1} \\ \mathbf{H}_h^{n+1} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \left\langle \begin{pmatrix} \mathbf{E}_h^{n} \\ \mathbf{H}_h^{n} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \cdots = \left\langle \begin{pmatrix} \mathbf{E}_h^{0} \\ \mathbf{H}_h^{0} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = 0,$$

where in the last step, we use

$$\left\langle \begin{pmatrix} \mathbf{E}_h^{0} \\ \mathbf{H}_h^{0} \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \left\langle \pi_h \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = \left\langle \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{pmatrix}, \begin{pmatrix} \nabla\psi \\ \nabla\phi \end{pmatrix} \right\rangle_{\mathbb{V}} = 0.$$

Therefore the conclusion of this proposition comes from taking $\phi = 0$ or $\psi = 0$, respectively. ∎

The mild version of the full discretization (5.1) can be rewritten as

$$u_n^{n+1} = S_{h,\tau} u_h^n - T_{h,\tau}\pi_h \Delta W^{n+1}, \tag{5.2}$$

where $T_{h,\tau} = \big(I - \frac{\tau}{2}M_h\big)^{-1}$ and $S_{h,\tau} = \big(I - \frac{\tau}{2}M_h\big)^{-1}\big(I + \frac{\tau}{2}M_h\big)$.

LEMMA 5.2. *For operators $T_{h,\tau}$ and $S_{h,\tau}$ on $\mathbb{V}_h$, the following estimates hold:*

(i) $\|T_{h,\tau}\|_{\mathcal{L}(\mathbb{V}_h,\mathbb{V}_h)} \le 1$.
(ii) $\|S^n_{h,\tau}\|_{\mathcal{L}(\mathbb{V}_h,\mathbb{V}_h)} \le 1$ *for any* $0 \le n \le N$.

Proof. To prove assertion (i), we define $\widetilde{v} = T_{h,\tau} v$ for any $v \in \mathbb{V}_h$, which means that $\widetilde{v} = v + \frac{\tau}{2} M_h \widetilde{v}$. Taking the inner product with $\widetilde{v}$ yields

$$\frac{1}{2}\Big[\|\widetilde{v}\|^2_{\mathbb{V}} - \|v\|^2_{\mathbb{V}} + \|\widetilde{v} - v\|^2_{\mathbb{V}}\Big] = \frac{\tau}{2}\langle M_h\widetilde{v}, \widetilde{v}\rangle_{\mathbb{V}} \le 0.$$

Hence $\|\widetilde{v}\|_{\mathbb{V}} = \|T_{h,\tau}v\|_{\mathbb{V}} \le \|v\|_{\mathbb{V}}$ leads to assertion (i).

Similarly, to prove assertion (ii), we define $v_h^n = S^n_{h,\tau} v$ for any $v \in \mathbb{V}_h$, which means that $v_h^\ell = v_h^{\ell-1} + \frac{\tau}{2}\big(M_h v_h^{\ell-1} + M_h v_h^\ell\big)$, $\ell = 1, 2, \dots, n$, with $v_h^0 = v$. Taking the inner product with $(v_h^{\ell-1} + v_h^\ell)$ yields $\|v_h^\ell\|^2_{\mathbb{V}} - \|v_h^{\ell-1}\|^2_{\mathbb{V}} \le 0$, and thus $\|v_h^\ell\|_{\mathbb{V}} \le \|v_h^{\ell-1}\|_{\mathbb{V}} \le \cdots \le \|v_h^0\|_{\mathbb{V}} = \|v\|_{\mathbb{V}}$. This leads to assertion (ii). ∎

PROPOSITION 5.3. *There exists a constant* C *independent of* h *and* τ *such that*

$$\max_{0\le n\le N} \mathbb{E}\|u_h^n\|^2_{\mathbb{V}} \le C(1 + \mathbb{E}\|u_0\|^2_{\mathbb{V}}).$$

Proof. From (5.2), we know that $u_h^n = S^n_{h,\tau}\pi_h u_0 - \sum_{j=1}^n S^{n-j}_{h,\tau} T_{h,\tau}\pi_h \Delta W^j$. Taking the $\|\cdot\|_{\mathbb{V}}$-norm on both sides of the above equation and using the triangle inequality, we get

$$\begin{aligned}\mathbb{E}\|u_h^n\|^2_{\mathbb{V}} &\le 2\mathbb{E}\|S^n_{h,\tau}\pi_h u_0\|^2_{\mathbb{V}} + 2\mathbb{E}\Big\|\sum_{j=1}^n S^{n-j}_{h,\tau} T_{h,\tau}\pi_h \Delta W^j\Big\|^2_{\mathbb{V}} \\ &\le 2\mathbb{E}\|\pi_h u_0\|^2_{\mathbb{V}} + 2\sum_{j=1}^n \mathbb{E}\left\|\pi_h \Delta W^j\right\|^2_{\mathbb{V}} \le 2\mathbb{E}\|u_0\|^2_{\mathbb{V}} + 2T\mathrm{Tr}(Q),\end{aligned}$$

which completes the proof. ∎

Let $W_{M;N,h} := \sum_{j=1}^N S^{N-j}_{h,\tau} T_{h,\tau}\pi_h \Delta W^j$. Then it is Gaussian on $\mathbb{V}_h$ with mean 0 and covariance operator

$$Q_{T;N,h} := \mathrm{Cov}(W_{M;N,h}) = \tau \sum_{j=1}^N \big(S^{N-j}_{h,\tau} T_{h,\tau}\pi_h\big) Q \big(S^{N-j}_{h,\tau} T_{h,\tau}\pi_h\big)^*.$$

Applying the fully discrete method to the small noise system (2.11), we denote by $\{\mathcal{L}\big(u_h^{N;u_0,\lambda}\big)\}_{\lambda>0}$ the laws of the full discretizations. The asymptotic behavior of $\{\mathcal{L}\big(u_h^{N;u_0,\lambda}\big)\}_{\lambda>0}$ is similar to that of $\{\mathcal{L}\big(u^{N;u_0,\lambda}\big)\}_{\lambda>0}$ in Proposition 3.8, which is stated below.

PROPOSITION 5.4. *For integer* $N > 0$ *and* $u_0 \in \mathbb{V}$*, the family of distributions* $\{\mathcal{L}\big(u_h^{N;\,u_0,\lambda}\big)\}_{\lambda>0}$ *satisfies the large deviation principle with the good rate function*

(5.3)
$$I^{u_0}_{T;N,h}(v) = \begin{cases} \frac{1}{2}\big\|\big(Q_{T;N,h}\big)^{-\frac{1}{2}}\big(v - S^N_{h,\tau}\pi_h u_0\big)\big\|^2_{\mathbb{V}}, & v - S^N_{h,\tau}\pi_h u_0 \in \big(Q_{T;N,h}\big)^{\frac{1}{2}}(\mathbb{V}), \\ +\infty, & \text{otherwise}. \end{cases}$$

5.1. Error estimate of full discretization. The error $u_h^n - u(t_n)$ is divided into $u_h^n - u(t_n) = (u_h^n - u^n) + (u^n - u(t_n))$, where the second term on the right-hand side is the error in temporal direction, which has been studied in Proposition 3.7.

Hence we only need to consider the error $u_h^n - u^n$. By inserting the term $\pi_h u^n$, we get $u_h^n - u^n = (u_h^n - \pi_h u^n) + (\pi_h u^n - u^n) =: e_h^n + e_\pi^n$.

Note that (4.2) and Propositions 3.3–3.4 yield that, for $k \in \{1,2\}$,

$$\left(\mathbb{E}\|e_\pi^n\|_{\mathbb{V}}^2\right)^{\frac{1}{2}} \leq Ch^k \left(\mathbb{E}\|u^n\|_{H^k(D)^6}^2\right)^{\frac{1}{2}} \leq Ch^k \left(1 + \mathbb{E}\|u_0\|_{\mathcal{D}(M_0^k)}^2\right)^{\frac{1}{2}}.$$

The estimate of error e_h^n is stated in the following theorem.

THEOREM 5.5. *Let $\{u^n,\ 0 \leq n \leq N\}$ in $L^2(\Omega; H^k(D))$ with $k \in \{1,2\}$ be the solution of* (3.1) *and let $\{u_h^n,\ 0 \leq n \leq N\}$ in $L^2(\Omega; \mathbb{V}_h)$ be the solution of* (5.1). *Then there is a constant C independent of h and τ such that*

$$\max_{0\leq n\leq N} \left(\mathbb{E}\|e_h^n\|_{\mathbb{V}}^2\right)^{\frac{1}{2}} \leq Ch^{k-\frac{1}{2}} \quad \textit{for } k \in \{1,2\}. \tag{5.4}$$

Proof. We apply the projection π_h to the temporal semidiscretization (3.1) and use Proposition 4.4(i) to get

$$\pi_h u^{n+1} = \pi_h u^n + \frac{\tau}{2}\left(M_h u^n + M_h u^{n+1}\right) - \pi_h \Delta W^{n+1}. \tag{5.5}$$

Subtracting (5.5) from (5.1) yields,

$$e_h^{n+1} = e_h^n + \frac{\tau}{2}\left(M_h e_h^n + M_h e_h^{n+1}\right) + \frac{\tau}{2}\left(M_h e_\pi^n + M_h e_\pi^{n+1}\right). \tag{5.6}$$

Applying $\langle \cdot, e_h^n + e_h^{n+1}\rangle_{\mathbb{V}}$, we obtain

(5.7)
$$\|e_h^{n+1}\|_{\mathbb{V}}^2 - \|e_h^n\|_{\mathbb{V}}^2 = \frac{\tau}{2}\langle M_h(e_h^n + e_h^{n+1}), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}} + \frac{\tau}{2}\langle M_h(e_\pi^n + e_\pi^{n+1}), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}}.$$

For the second term on the right-hand side of (5.7), we use (4.12) to get

$$\begin{aligned}\langle M_h(e_\pi^n + e_\pi^{n+1}), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}} \leq &-\frac{1}{2}\langle M_h(e_h^n + e_h^{n+1}), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}} \\ &+ Ch^{2k-1}\|u^n + u^{n+1}\|_{H^k(D)^6}^2.\end{aligned}$$

Hence (5.7) becomes

$$\|e_h^{n+1}\|_{\mathbb{V}}^2 - \|e_h^n\|_{\mathbb{V}}^2 \leq \frac{\tau}{4}\langle M_h\left(e_h^n + e_h^{n+1}\right), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}} + C\tau h^{2k-1}\|u^n + u^{n+1}\|_{H^k(D)^6}^2.$$

Proposition 4.4(ii) leads to $\langle M_h(e_h^n + e_h^{n+1}), e_h^n + e_h^{n+1}\rangle_{\mathbb{V}} \leq 0$, and thus

$$\mathbb{E}\|e_h^{n+1}\|_{\mathbb{V}}^2 - \mathbb{E}\|e_h^n\|_{\mathbb{V}}^2 \leq C\tau h^{2k-1}\mathbb{E}\left(\|u^n\|_{H^k(D)^6}^2 + \|u^{n+1}\|_{H^k(D)^6}^2\right) \leq C\tau h^{2k-1}.$$

Gronwall's inequality yields the conclusion. ∎

Combining error estimates in temporal and spatial directions, we finally obtain the error estimate for the full discretization (5.1).

THEOREM 5.6. *If the assumptions of Theorems* 5.5 *and* 3.7 *are satisfied, then the fully discrete error $u_h^n - u(t_n)$ is bounded by*

$$\left(\mathbb{E}\|u_h^n - u(t_n)\|_{\mathbb{V}}^2\right)^{\frac{1}{2}} \leq C\tau^{\frac{k}{2}} + Ch^{k-\frac{1}{2}} \quad \textit{for } k \in \{1,2\},$$

where the constant C is independent of h and τ.

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